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ANDRE OSTRAK

Diameter two properties in spaces of Lipschitz functions

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Diameter two properties in spaces of Lipschitz functions

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Institute of Mathematics and Statistics, Faculty of Science and Technology, University of Tartu, Estonia.

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Supervisors:

Rainis Haller
University of Tartu
Tartu, Estonia

Märt Põldvere
University of Tartu
Tartu, Estonia

Opponents:

Ramón J. Aliaga
Polytechnic University of Valencia
Valencia, Spain

Antonín Procházka
University of Franche-Comté
Besançon, France

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Chapter 1

Introduction

1.1 Background

Some geometric phenomena of Banach spaces appear only in the infinite-dimensional spaces. For example, there exist Banach spaces where every slice of the unit ball has a diameter of 2; such spaces are said to have the slice diameter 2 property (briefly, slice-D2P). Properties which imply that all specific subsets of the unit ball, e.g., slices, nonempty relatively weakly open subsets, and finite convex combinations of slices, have diameter 2 are usually referred to as diameter 2 properties. Banach spaces with diameter 2 properties reside on the other end of the spectrum from those with the Radon–Nikodým property. This is because the unit ball of a Banach space with the Radon–Nikodým property has slices of arbitrarily small diameter. All reflexive Banach spaces, including all finite-dimensional spaces, have the Radon–Nikodým property.

Moving even further away from the Radon–Nikodým property, we find the diameter 2 property (briefly, D2P), the strong diameter 2 property (briefly, SD2P), and the symmetric strong diameter 2 property (briefly, SSD2P). A Banach space is said to have the D2P if every nonempty relatively weakly open subset of the unit ball has diameter 2, it is said to have the SD2P if every convex combination of slices of the unit ball has diameter 2, and it is said to have the SSD2P if, given a finite number of slices of the unit ball, there exists a direction such that all these slices contain a line segment of length almost 2 in this direction. If a Banach space is a dual space, then we also consider the weak* versions of these diameter two properties (w^*-slice-D2P, w^*-D2P, w^*-SD2P, and w^*-SSD2P), where slices and weakly open subsets in the above definitions are replaced by weak* slices and weak* open subsets, respectively.

It was shown in [3] that every Banach space with the SSD2P has the SD2P. An immediate consequence of Bourgain's Lemma is that every Banach space with the SD2P has the D2P. A Banach space with the D2P has the slice-D2P because slices of the unit ball are relatively weakly open subsets of the unit ball.

To the best of our knowledge, the study of the diameter 2 properties was started in [30] by Nygaard and Werner, who showed that nonempty relatively weakly open subsets of the unit ball in uniform algebras have diameter 2, meaning that the uniform algebras have the D2P. Later, it was shown in [7] that all M-embedded spaces have the D2P. A more systematic treatment of diameter 2 properties can be found in [3] where also a survey of previous results regarding this topic was given.

One of the most well-studied diameter 2 properties was derived from a result by Daugavet, who showed in [11] that every compact operator T on $C[0,1]$ satisfies the equality $\|I + T\| = 1 + \|T\|$, where I is the identity operator. This equality is also known as the Daugavet equation. Other examples of Banach spaces, for which the Daugavet equation holds for all compact operators, followed shortly, e.g., $L_1[0,1]$ (see [29]). A Banach space is said to have the Daugavet property if every rank 1 operator on the space satisfies the Daugavet equation. It is known (see [35]) that a Banach space with the Daugavet property has the SD2P. However, there exist Banach spaces with the Daugavet property but without the SSD2P and vice versa.

A Banach space has the SD2P if and only if its dual space is octahedral (see [12] and [15], see also [8]). In [21], similar dual characterisations were given for the D2P and the slice-D2P. A similar dual characterisation for the SSD2P is currently unknown.

In this thesis, we study diameter 2 properties in the spaces of Lipschitz functions with an emphasis on the (w^*-)SSD2P. One of the first papers to appear on this topic was by Ivakhno, who showed in [23] that the space of Lipschitz functions has the slice-D2P if the underlying metric space is unbounded, not uniformly discrete, or K_n with $n = 2$ or $n = 3$. The latter space K_n is the metric subspace of ℓ_∞ where the terms of the sequences are nonnegative integers not greater than n. In [34], Procházka and Rueda Zoca showed that the space of Lipschitz functions has the w^*-SD2P if and only if the underlying metric space has a property they named the long trapezoid property. In [20], the w^*-SSD2P was studied in the spaces of Lipschitz functions. In that paper, the authors asked whether every space of Lipschitz functions with the w^*-SD2P has even the w^*-SSD2P. In [10], it was shown that the space of Lipschitz functions has the SSD2P whenever the underlying metric space has infinitely many limit points or if it is discrete but not uniformly discrete. A significant result in [13] gives a characterisation for the Daugavet property in the spaces of Lipschitz functions. Namely, it was shown that the space of Lipschitz functions has the Daugavet property if and only if the underlying metric space is a length space.

1.2 Summary of the thesis

The main aim of this thesis is to investigate diameter 2 properties in spaces of Lipschitz functions with an emphasis on the (w^*-)SSD2P. A characterisation of the w^*-SSD2P will be given for the space $\mathrm{Lip}_0(M)$ in terms of the underlying metric space M and also in terms of the Lipschitz-free space $\mathcal{F}(M)$. The diameter 2 property, the strong diameter 2 property, and the symmetric strong diameter 2 property are shown to be different for the spaces of Lipschitz functions.

The thesis consists of four chapters which are organised as follows.

Chapter 1 gives historical background and a summary of the thesis, followed by the notation and preliminaries used throughout the thesis.

In Chapter 2, we characterise the w^*-SSD2P for $\mathrm{Lip}_0(M)$ in terms of the metric of the underlying space M. We apply this characterisation to show that the w^*-SSD2P is different from the w^*-SD2P for spaces of Lipschitz functions, thereby answering a question posed in [20, Question 6.3]. This chapter is mainly based on [32].

In Chapter 3, we characterise the w^*-SSD2P for $\mathrm{Lip}_0(M)$ by a property of its predual, the Lipschitz-free space $\mathcal{F}(M)$. We call this new property decomposable octahedrality and study its duality with the SSD2P in the general context of Banach spaces. We show that, for a Banach space to be decomposably octahedral, it is sufficient that its dual space has the w^*-SSD2P. We give necessary and sufficient conditions for the absolute sum of two Banach spaces to be decomposably octahedral and show that the space $C(K)$ of all continuous functions on a compact Hausdorff space K is never decomposably octahedral. This chapter is mainly based on [33].

In Chapter 4, we solve some open problems regarding diameter 2 properties in spaces of Lipschitz functions by using the de Leeuw's transform. We show that the D2P, the SD2P, and the SSD2P are all different properties for the spaces of Lipschitz functions and that the space $\mathrm{Lip}_0(K_n)$ has the SSD2P for every $n \in \mathbb{N}$; these results answer two questions posed in [28]. We also show that every local norm-one Lipschitz function is a Daugavet point, thereby answering a question posed in [25]. This chapter is mainly based on [22].

1.3 Notation

In this thesis, we use standard Banach space notation.

We consider only nontrivial Banach spaces over the field of real numbers. Given a Banach space X, we denote the closed unit ball, the unit sphere, and the dual space of X by B_X, S_X, and X^*, respectively. For a subset A of Banach space X, we denote the linear span and the closed linear span of A by $\operatorname{span} A$ and $\overline{\operatorname{span}} A$, respectively.

Given a metric space M, a point x in M, and $r \geqslant 0$, we denote by $B(x, r)$ the open ball in M centred at x of radius r (we use the convention $B(x, r) = \varnothing$ if $r = 0$).

1.4 Preliminaries

We introduce some main notions and results used throughout the thesis. We start by giving an overview of diameter 2 properties and their dual notions in Banach spaces.

Diameter 2 properties in Banach spaces

Let X be a Banach space.

Definition 1.1. A *slice* of B_X is a set of the form

$$S(x^*, \alpha) \coloneqq \{x \in B_X : x^*(x) > 1 - \alpha\}$$

where $x^* \in S_{X^*}$ and $\alpha > 0$.

If X is a dual space, then slices whose defining functional comes from (the canonical image of) the predual of X are called *weak* slices*.

Definition 1.2 ([3] and [4], see also [9])**.** A Banach space X is said to have the

(1) *slice diameter* 2 *property* (briefly, *slice-D*2*P*) if every slice of B_X has diameter 2;

(2) *diameter* 2 *property* (briefly, *D*2*P*) if every nonempty relatively weakly open subset of B_X has diameter 2;

(3) *strong diameter* 2 *property* (briefly, *SD*2*P*) if every convex combination of slices of B_X has diameter 2, i.e., the diameter of $\sum_{i=1}^{n} \lambda_i S_i$ is 2 whenever $n \in \mathbb{N}$, $\lambda_1, \ldots, \lambda_n \geqslant 0$ with $\sum_{i=1}^{n} \lambda_i = 1$, and $S_1, \ldots, S_n$ are slices of B_X;

(4) *symmetric strong diameter 2 property* (briefly, *SSD2P*) if, for every $n \in \mathbb{N}$, every family $\{S_1, \ldots, S_n\}$ of slices of B_X, and every $\varepsilon > 0$, there exist $f_1 \in S_1, \ldots, f_n \in S_n$, and $g \in B_X$ with $\|g\| > 1 - \varepsilon$ such that $f_i \pm g \in S_i$ for every $i \in \{1, \ldots, n\}$.

The following implications hold for these properties:

$$\text{SSD2P} \Longrightarrow \text{SD2P} \Longrightarrow \text{D2P} \Longrightarrow \text{slice-D2P}$$

The third implication holds because every slice of B_X is a relatively weakly open subset of B_X. The second implication is a consequence of Bourgain's Lemma (see, e.g., [14, Lemma II.1]), which, in particular, says that every nonempty relatively weakly open subset of B_X contains a convex combination of slices of B_X. The first implication was proved in [3, Lemma 4.1].

The reverse implications do not hold in general, e.g., the space $c_0 \oplus_2 c_0$ has the D2P but not the SD2P (see, e.g., [5], [19], or [31]); a renorming of c_0 with the D2P but without the slice-D2P was constructed in [9]; the space $L_1[0,1]$ has the SD2P but not the SSD2P (see [4], for a proof see [20, Theorem 3.1 and Remark 3.3]).

If X is a dual space and the slices and weakly open subsets in Definition 1.2 are replaced by weak* slices and weak* open subsets, respectively, then we consider the weak* versions of the defined properties (*w^*-slice-D2P*, *w^*-D2P*, *w^*-SD2P*, and *w^*-SSD2P*). It is easy to see that if a dual Banach space has an above-mentioned (non-weak*) diameter 2 property then it also has the corresponding weak* diameter 2 property. The reverse implication does not hold in general. In [21], it was shown that the dual space of $C[0,1]$ has the w^*-SD2P but not the slice-D2P. Whether there exists a dual Banach space which has the w^*-SSD2P but not the SSD2P is currently unknown to us.

By similar reasoning to the one after Definition 1.2, it can be shown that the following implications hold:

$$w^*\text{-SSD2P} \Longrightarrow w^*\text{-SD2P} \Longrightarrow w^*\text{-D2P} \Longrightarrow w^*\text{-slice-D2P}$$

Again, the reverse implications do not hold in general. This is due to the known fact that a Banach space X has the slice-D2P (respectively, D2P, SD2P, SSD2P) if and only if its bidual X^{**} has the w^*-slice-D2P (respectively, w^*-D2P, w^*-SD2P, w^*-SSD2P).

Some diameter 2 properties have a useful dual characterisation.

Definition 1.3 (see [12] and [21])**.** A Banach space X is said to be *octahedral* (briefly, *OH*) if, whenever $n \in \mathbb{N}$, $x_1, \ldots, x_n \in S_X$, and $\varepsilon > 0$, there exists a $y \in S_X$ such that

$$\|x_i + y\| > 2 - \varepsilon.$$

It is known (see [12] and [15], for a detailed proof see, e.g., [8] or [21]) that a dual Banach space X^* has the w^*-SD2P if and only if X is OH. The following properties were introduced in [21] to give a similar predual characterisation for the w^*-D2P and the w^*-slice-D2P.

Definition 1.4 (see [21, Definition 2.2])**.** A Banach space X is said to be

(1) *locally octahedral* (briefly, *LOH*) if, for every $x \in S_X$ and every $\varepsilon > 0$, there exists a $y \in S_X$ such that

$$\|x \pm y\| \geqslant 2 - \varepsilon;$$

(2) *weakly octahedral* (briefly, *WOH*) if, for every finite-dimensional subspace E of X, every $x^* \in B_{X^*}$, and every $\varepsilon > 0$, there exists a $y \in S_X$ such that, for all $x \in E$,

$$\|x + y\| \geqslant (1 - \varepsilon)\big(|x^*(x)| + \|y\|\big).$$

In [21, Theorem 3.1], it was shown that X^* has the w^*-slice D2P if and only if X is LOH. In [21, Theorem 3.3], it was shown that X^* has the w^*-D2P if and only if X is WOH. It follows that

(1) X has the SD2P if and only if X^* is OH;

(2) X has the D2P if and only if X^* is WOH;

(3) X has the slice-D2P if and only if X^* is LOH.

Clearly

$$\text{OH} \Longrightarrow \text{WOH} \Longrightarrow \text{LOH},$$

and the reverse implications do not hold in general.

One of the most well-studied diameter 2 properties is the following.

Definition 1.5. A Banach space X is said to have the *Daugavet property* if, for every rank 1 operator $T\colon X \to X$, one has

$$\|I + T\| = 1 + \|T\| \tag{1.1}$$

where I denotes the identity operator.

This property stems from a remarkable result by Daugavet, who showed in [27] that the Banach space $C[0,1]$ has this property. In fact, the result by Daugavet stated that the equality (1.1), also known as the Daugavet equation, holds for every compact operator T. It was shown in [26] that if every rank 1 operator satisfies the equation, then so does every weakly compact operator. Other examples of Banach spaces with the Daugavet property followed shortly after Daugavet's paper, e.g., $L_1[0,1]$ (see [29]). The following geometric characterisation of the Daugavet property was given by Werner in [37]: X has the Daugavet property if and only if, for every $x \in S_X$, every slice S of B_X, and every $\varepsilon > 0$, there exists a $y \in S$ such that $\|x - y\| \geqslant 2 - \varepsilon$. This characterisation was the inspiration behind the following notion: an element $x \in S_X$ is a *Daugavet-point*, if, given a slice S of B_X and an $\varepsilon > 0$, there exists a $y \in S$ with $\|x - y\| > 2 - \varepsilon$ (see [1]). A Banach space with the Daugavet property has the SD2P and is also OH (see [35, proof of Lemma 3], see also [3, Theorem 4.4]). The Daugavet property does not imply the SSD2P because, as previously stated, the space $L_1[0,1]$ does not have the SSD2P.

The study of diameter 2 properties is closely related to that of almost square Banach spaces.

Definition 1.6 ([2, Definition 1.1])**.** A Banach space X is said to be

(1) *locally almost square* (briefly, *LASQ*) if, for every $x \in S_X$ and every $\varepsilon > 0$, there exists a $y \in S_X$ such that

$$\|x \pm y\| \leqslant 1 + \varepsilon;$$

(2) *weakly almost square* (briefly, *WASQ*) if, for every $x \in S_X$, there exists a sequence (y_n) in S_X such that $\|x \pm y_n\| \to 1$, $\|y_n\| \to 1$, and $y_n \to 0$ weakly;

(3) *almost square* (briefly, *ASQ*) if, for every finite subset $\{x_1, \ldots, x_n\}$ of S_X and every $\varepsilon > 0$, there exists a $y \in S_X$ such that

$$\|x_i \pm y\| \leqslant 1 + \varepsilon$$

for every $i \in \{1, \ldots, n\}$.

In [2, Theorem 2.8], it was shown that every ASQ Banach space is WASQ. The reverse implication does not hold in general, for instance, the space $L_1[0,1]$ (more generally, every Cesàro function space) is WASQ but not ASQ (see [2, Example 3.3]). Every WASQ Banach space is clearly LASQ. Whether the converse of the latter statement holds is currently unknown to us.

In [27], it was shown that every LASQ Banach space has the slice-D2P, and that every WASQ Banach space has the D2P. In [2], it was shown that every ASQ Banach space has the SD2P. In fact, by [4], every ASQ Banach space even has the SSD2P. On the other hand, the Banach space $C[0,1]$ has the SSD2P (see, e.g., [4]) but is not LASQ (to see this, check the condition for the constant function 1).

The relations between the above-mentioned diameter 2 properties and almost square spaces can be summed up with the following diagram:

$$\begin{tikzcd}
 & \text{ASQ} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{WASQ} \arrow[r, Rightarrow] \arrow[dd, Rightarrow] & \text{LASQ} \arrow[dd, Rightarrow] \\
 & \text{SSD2P} \arrow[d, Rightarrow] & & \\
\begin{array}{c}\text{Daugavet}\\ \text{property}\end{array} \arrow[r, Rightarrow] & \text{SD2P} \arrow[r, Rightarrow] & \text{D2P} \arrow[r, Rightarrow] & \text{slice-D2P}
\end{tikzcd}$$

Spaces of Lipschitz functions and Lipschitz-free spaces

The main aim of this thesis is to study diameter 2 properties in the spaces of Lipschitz functions with an emphasis on the (w^*-)SSD2P.

Recall that, given metric spaces (M, d_M) and (N, d_N), a function $f\colon M \to N$ is a *Lipschitz function* if there exists an $L \geqslant 0$ such that, for all $x, y \in M$ with $x \neq y$, one has

$$d_N\big(f(x), f(y)\big) \leqslant L\, d_M(x, y);$$

the smallest such L is called the *Lipschitz constant* of f.

Let M be a *pointed metric space*, that is, a metric space with a fixed point 0. The *space of Lipschitz functions* $\mathrm{Lip}_0(M)$ is the Banach space of all Lipschitz functions $f\colon M \to \mathbb{R}$ with $f(0) = 0$ equipped with the norm

$$\|f\| = \sup\Big\{\frac{|f(x) - f(y)|}{d(x, y)} : x, y \in M,\ x \neq y\Big\},$$

i.e., $\|f\|$ is the Lipschitz constant of f.

It is known that $\mathrm{Lip}_0(M)$ is a dual space whose predual, the *Lipschitz-free space*, also known as the *Arens–Eells space*, can be defined as the subspace $\mathcal{F}(M) = \overline{\operatorname{span}}\{\delta_x\colon x \in M\}$ of $\mathrm{Lip}_0(M)^*$ where $\langle f, \delta_x\rangle = f(x)$ for every $f \in \mathrm{Lip}_0(M)$. The mapping $\delta\colon M \to \mathcal{F}(M)$,

$$\delta(x) = \delta_x \qquad \text{for all } x \in M,$$

is an isometric embedding. The Lipschitz-free space is uniquely characterised (up to linear isometry) by the following universal property.

Proposition 1.7. *Let M be a pointed metric space. For every Banach space X and every Lipschitz function $f\colon M \to X$ with $f(0) = 0$, there exists a unique linear mapping $\widetilde{f}\colon \mathcal{F}(M) \to X$ such that $\widetilde{f} \circ \delta = f$ and $\|\widetilde{f}\|$ is equal to the Lipschitz constant of f.*

Using this universal property, one can see that Lipschitz-free spaces provide a canonical linearisation of Lipschitz functions between (pointed) metric spaces. More precisely, let M and N be pointed metric spaces and let $f\colon M \to N$ be a Lipschitz function with $f(0) = 0$, then there exists a unique linear map $\widetilde{f}\colon \mathcal{F}(M) \to \mathcal{F}(N)$ such that $\|\widetilde{f}\|$ is equal to the Lipschitz constant of f and the following diagram commutes:

$$\begin{tikzcd}
\mathrm{M} \arrow[r, "f"] \arrow[d, "\delta_M"'] & \mathrm{N} \arrow[d, "\delta_N"] \\
\mathcal{F}(M) \arrow[r, "\widetilde{f}"] & \mathcal{F}(N)
\end{tikzcd}$$

For $x, y \in M$, it is known that $\|\delta_x - \delta_y\| = d(x,y)$. If $x, y \in M$ with $x \neq y$, then we will denote by $m_{x,y}$ the norm-one element $\frac{\delta_x - \delta_y}{d(x,y)}$; we will often refer to such elements as *elementary molecules*. It is not hard to verify that if $\mu \in \operatorname{span}\{\delta_x\colon x \in M\}$, then

$$\|\mu\| = \inf\Big\{\sum_{i=1}^{n} \lambda_i\colon \mu = \sum_{i=1}^{n} \lambda_i m_{x_i,y_i},\ x_i, y_i \in M,\ x_i \neq y_i,\ n \in \mathbb{N}\Big\}$$

where the infimum is taken over all expressions of μ as a linear combination of elementary molecules $m_{x,y}$.

An important and well-known result in the theory of Lipschitz functions is McShane's extension theorem which says that if N is a subset of M and $f\colon N \to \mathbb{R}$ is a Lipschitz function with the Lipschitz constant L, then there is an extension to a Lipschitz function $M \to \mathbb{R}$ with the same Lipschitz constant L. Moreover, there exists the smallest and the greatest such extensions $\check{f}$ and $\hat{f}$, respectively; they are given by the formulae

$$\check{f}(y) = \sup\{f(x) - L\, d(x,y) : x \in N\} \qquad \text{for all } y \in M$$

and

$$\hat{f}(y) = \inf\{f(x) + L\, d(x,y) : x \in N\} \qquad \text{for all } y \in M.$$

For a thorough treatment of spaces of Lipschitz functions and Lipschitz-free spaces, we refer the reader to [16], [17], and [36].

The diameter 2 properties of spaces of Lipschitz functions have been studied in a number of papers. In [23], Ivakhno proved that if a metric space M is unbounded or not uniformly discrete, or $M = K_2$ or $M = K_3$, then the space $\mathrm{Lip}_0(M)$ has the slice-D2P. Recall that M is said to be *uniformly discrete* if $\inf\{d(x,y)\colon x,y \in M,\ x \neq y\} > 0$. The space K_n is the metric subspace of ℓ_∞ where the terms of the sequences are nonnegative integers not greater than n. In [34], Procházka and Rueda Zoca characterised the octahedrality of the Lipschitz-free space $\mathcal{F}(M)$ by the following property of the metric space M.

Definition 1.8. A metric space M is said to have the *long trapezoid property* (briefly, *LTP*) if, for every $\varepsilon > 0$ and every finite subset N of M, there exist elements $u, v \in M$ with $u \neq v$ satisfying, for all $x, y \in N$,

$$(1-\varepsilon)\big(d(x,y) + d(u,v)\big) \leqslant d(x,u) + d(y,v).$$

More precisely, they showed that the following theorem holds.

Theorem 1.9 ([34, Theorem 3.1])**.** *Let M be a pointed metric space. The following statements are equivalent:*

(i) $\mathrm{Lip}_0(M)$ *has the w^*-SD2P;*

(ii) $\mathcal{F}(M)$ *is OH;*

(iii) M *has the LTP.*

The Daugavet property in the spaces of Lipschitz functions and Lipschitz-free spaces has been extensively studied. The following is a recent result by Avilés and Martínez-Cervantes.

Theorem 1.10. *Let M be a complete metric space. Then the following assertions are equivalent:*

(i) M *is length;*

(ii) $\mathrm{Lip}_0(M)$, *i.e., $\mathcal{F}(M)^*$, has the Daugavet property;*

(iii) $\mathcal{F}(M)$ *has the Daugavet property;*

(iv) $\mathcal{F}(M)$ *has the strong diameter 2 property;*

(v) $\mathcal{F}(M)$ *has the diameter 2 property;*

(vi) $\mathcal{F}(M)$ *has the slice diameter 2 property;*

(vii) *the unit ball of $\mathcal{F}(M)$ does not have strongly exposed points;*

(viii) *M has property (Z).*

This result appears in [6, Theorem 1.5]. In fact, the main contribution of [6] was the implication (viii)⇒(i). The implications (iii)⇒(i) and (vii)⇔(viii) were already obtained in [13], and (i)⇒(ii) and (i)⇒(viii) were obtained in [24]. The implications (ii)⇒(iii)⇒(iv)⇒(v)⇒(vi)⇒(viii) hold in the more general setting when $\mathcal{F}(M)$ is replaced with any Banach space.

It was shown in a recent preprint that the Lipschitz-free space $\mathcal{F}(M)$ is LASQ if and only if M is length, but that $\mathcal{F}(M)$ is never ASQ [18, Theorems 3.1 and 4.1]. It is unknown to us if the Lipschitz-free space $\mathcal{F}(M)$ is WASQ whenever M is length. It is also unknown to us if there are Lipschitz-free spaces with the SSD2P.

Chapter 2

Weak-star symmetric strong diameter two property in spaces of Lipschitz functions

The symmetric strong diameter 2 property first appeared in [3] but was studied more extensively in [4], [21], [10], and [28]. We give a brief overview of the results pertaining to our study of the property for the space $\mathrm{Lip}_0(M)$ in terms of the underlying metric space M. This chapter is based on [32].

Let M be a pointed metric space. It is well known that a dual Banach space X^* has the w^*-SD2P if and only if (the norm of) X is octahedral ([12], [15], for a proof, see, e.g., [8] or [21]). Therefore, the space $\mathrm{Lip}_0(M)$ has the w^*-SD2P if and only if the norm of $\mathcal{F}(M)$ is octahedral. Recall that a metric space M is said to have the *long trapezoid property* (briefly, *LTP*) if, for every finite subset N of M and every $\varepsilon > 0$, there exist elements $u, v \in M$ with $u \neq v$ satisfying, for all $x, y \in N$,

$$(1-\varepsilon)\big(d(x,y) + d(u,v)\big) \leqslant d(x,u) + d(y,v).$$

In [34, Theorem 3.1], it was shown that the norm of $\mathcal{F}(M)$ is octahedral if and only if the metric space M has the LTP. Therefore, the space $\mathrm{Lip}_0(M)$ has the w^*-SD2P if and only if M has the LTP. In this chapter, we give a similar characterisation to the weak* symmetric strong diameter 2 property.

Definition 2.1. A dual Banach space X^* is said to have the *weak* symmetric strong diameter* 2 *property* (briefly, *w^*-SSD2P*) if, for every $n \in \mathbb{N}$, every family $\{S_1, \ldots, S_n\}$ of w^*-slices of B_X, and every $\varepsilon > 0$, there exist $f_1 \in S_1, \ldots, f_n \in S_n$, and $g \in B_{X^*}$ with $\|g\| > 1-\varepsilon$ such that $f_i \pm g \in S_i$ for every $i \in \{1, \ldots, n\}$.

It is known that in general the w^*-SSD2P is a strictly stronger property than the w^*-SD2P (see, e.g., [20]). In [20, Theorem 5.7] it was shown that the space $\mathrm{Lip}_0(M)$ has the w^*-SSD2P if at least one of the following conditions holds for space M:

(1) M is unbounded;

(2) M is not uniformly discrete;

(3) M is a discrete metric space;

(4) $M = K_n$, where $n \in \mathbb{N}$ and the space K_n is the metric subspace of ℓ_∞ where the terms of the sequences are from the set $\{0, 1, \ldots, n\}$.

However, it remained open whether the w^*-SD2P and the w^*-SSD2P are indeed different for the spaces of Lipschitz functions [20, Question 6.3]. We answer this question by defining a metric space M for which the space $\mathrm{Lip}_0(M)$ has the w^*-SD2P but not the w^*-SSD2P. Moreover, we characterise the w^*-SSD2P for the space $\mathrm{Lip}_0(M)$ in terms of the metric of the underlying space M. More precisely, we prove Theorem 2.3, which says that $\mathrm{Lip}_0(M)$ has the w^*-SSD2P if and only if M enjoys the following property.

Definition 2.2. We say that a metric space M has the *strong long trapezoid property* (briefly, *SLTP*) if, for every finite subset N of M and every $\varepsilon > 0$, there exist $u, v \in M$ with $u \neq v$ satisfying, for all $x, y \in N$,

$$(1-\varepsilon)\big(d(x,y) + d(u,v)\big) \leqslant d(x,u) + d(y,v). \tag{2.1}$$

and, for all $x, y, z, w \in N$,

$$\begin{aligned}(1-\varepsilon)\big(2d(u,v) + d(x,y) + d(z,w)\big)&\\ \leqslant d(x,u) + d(y,u) + d(z,v) + d(w,v).&\end{aligned} \tag{2.2}$$

We then apply Theorem 2.3 to show that, for spaces of Lipschitz functions, the w^*-SSD2P is a strictly stronger property than the w^*-SD2P: Examples 2.5 and 2.6 provide metric spaces which have the LTP but not the SLTP.

A question that arises from the definition of the SLTP is whether the inequality (2.2) implies (2.1). Example 2.7 shows that this is not the case: it provides a metric space M for which (2.2) holds for every finite subset N, but which fails the LTP.

We finish the chapter by showing that any infinite subset of ℓ_1, viewed as a metric space, has the SLTP (Example 2.8).

2.1 Main result

Theorem 2.3. *Let M be a pointed metric space. The following statements are equivalent:*

(i) $\mathrm{Lip}_0(M)$ *has the* w^**-SSD2P;*

(ii) M *has the SLTP.*

For part (ii)$\Rightarrow$(i) of the theorem it is convenient to use the following lemma, which we will also make use of in Chapter 4 in the proof of Theorem 4.7.

Lemma 2.4. *Let N be a subset of M, let $\varepsilon > 0$, and let $h_1, \ldots, h_n \in \mathrm{Lip}_0(M)$ with $\|h_i\| \leqslant 1 - \varepsilon$ for every $i \in \{1, \ldots, n\}$. If there exist $u, v \in M$ with $u \neq v$ satisfying inequalities (2.1) and (2.2) for all $x, y \in N$, then there exist functions $f_1, \ldots, f_n, g \in \mathrm{Lip}_0(M)$ satisfying*

- $f_i|_N = h_i|_N$ *for every* $i \in \{1, \ldots, n\}$*;*
- $g|_N = 0$ *and* $\|g\| \geqslant 1 - \varepsilon$*;*
- $\|f_i \pm g\| \leqslant 1$ *for every* $i \in \{1, \ldots, n\}$*.*

Proof. Assume that there exist $u, v \in M$ with $u \neq v$, satisfying the inequalities (2.1) and (2.2) for all $x, y, z, w \in N$. Setting

$$r_0 = \frac{1}{2} \inf_{x,y \in N} \big(d(x,u) + d(y,u) - (1-\varepsilon) d(x,y)\big)$$

and

$$s_0 = \frac{1}{2} \inf_{z,w \in N} \big(d(z,v) + d(w,v) - (1-\varepsilon) d(z,w)\big),$$

one has $r_0 + s_0 \geqslant (1-\varepsilon) d(u,v)$. Thus, there exist $r, s \geqslant 0$ with $r \leqslant r_0$ and $s \leqslant s_0$ such that

$$r + s = (1-\varepsilon) d(u,v).$$

We may assume that $r > 0$. Define a function $g \colon M \to \mathbb{R}$ by

$$g(x) = \begin{cases} r - d(x,u) & \text{if } x \in B(u,r); \\ -s + d(x,v) & \text{if } x \in B(v,s); \\ 0 & \text{otherwise.} \end{cases}$$

Set $B = B(u,r) \cup B(v,s)$ and observe that $g|_N = 0$ because $N \cap B = \varnothing$. One has $\|g\| \leqslant 1$ (here we use that, whenever $x \in B(u,r)$ and $y \in B(v,s)$, one has $g(y) \leqslant 0 \leqslant g(x)$, and thus $|g(x)-g(y)| = g(x)-g(y)$). One also has $\|g\| \geqslant (1-\varepsilon)$, because

$$|g(u) - g(v)| = g(u) - g(v) = r + s = (1-\varepsilon)d(u,v).$$

Fix $i \in \{1,\dots,n\}$. We first define f_i on the set $L = N \cup B$. Let $f_i|_N = h_i|_N$. We next show that there is a $c_i \in \mathbb{R}$ such that, by defining $f_i|_B = c_i$, one has $\|f_i \pm g\|_{\mathrm{Lip}_0(L)} \leqslant 1$ and $\|f_i \pm |g|\|_{\mathrm{Lip}_0(L)} \leqslant 1$.

Set

$$\begin{aligned}
\breve{a}_i &= \sup_{x\in N}\big(h_i(x) - d(x,u)\big), & \widehat{a}_i &= \inf_{x\in N}\big(h_i(x) + d(x,u)\big),\\
\breve{b}_i &= \sup_{x\in N}\big(h_i(x) - d(x,v)\big), & \widehat{b}_i &= \inf_{x\in N}\big(h_i(x) + d(x,v)\big).
\end{aligned}$$

Whenever $x, y \in N$, since $\|h_i\| \leqslant 1-\varepsilon$, one has

$$h_i(x) + d(x,u) - \big(h_i(y) - d(y,u)\big) \geqslant d(x,u) + d(y,u) - (1-\varepsilon)d(x,y) \geqslant 2r,$$

and, by (2.1),

$$\begin{aligned}
h_i(x) + d(x,u) - \big(h_i(y) - d(y,v)\big) &\geqslant d(x,u) + d(y,v) - (1-\varepsilon)d(x,y)\\
&\geqslant (1-\varepsilon)d(u,v) = r + s.
\end{aligned}$$

Thus, $\widehat{a}_i - r \geqslant \breve{a}_i + r$ and $\widehat{a}_i - r \geqslant \breve{b}_i + s$. Similarly, one observes that $\widehat{b}_i - s \geqslant \breve{b}_i + s$ and $\widehat{b}_i - s \geqslant \breve{a}_i + r$. It follows that there exists a $c_i \in \big[\breve{a}_i + r, \widehat{a}_i - r\big] \cap \big[\breve{b}_i + s, \widehat{b}_i - s\big]$. This c_i does the job.

Indeed, let $x \in N$ and $y \in B(u,r)$. In order to see that

$$\big|f_i(x) \pm g(x) - \big(f_i(y) \pm g(y)\big)\big| = \Big|h_i(x) - \Big(c_i \pm \big(r - d(y,u)\big)\Big)\Big| \leqslant d(x,y),$$

it suffices to show that

$$h_i(x) - d(x,y) \pm d(y,u) \leqslant c_i \pm r \leqslant h_i(x) + d(x,y) \pm d(y,u). \tag{2.3}$$

These inequalities hold:

$$\begin{aligned}
h_i(x) - d(x,y) - d(y,u) &\leqslant h_i(x) - d(x,u)\\
&\leqslant \breve{a}_i \leqslant c_i - r \leqslant \widehat{a}_i - 2r\\
&\leqslant h_i(x) + d(x,u) - 2d(y,u)\\
&\leqslant h_i(x) + d(x,y) - d(y,u)
\end{aligned}$$

and

$$\begin{aligned} h_i(x) - d(x,y) + d(y,u) &\leqslant h_i(x) - d(x,u) + 2d(y,u) \\ &\leqslant \check{a}_i + 2r \leqslant c_i + r \leqslant \widehat{a}_i \\ &\leqslant h_i(x) + d(x,u) \\ &\leqslant h_i(x) + d(x,y) + d(y,u). \end{aligned}$$

The inequalities

$$\left| f_i(x) \pm |g(x)| - \big(f_i(y) \pm |g(y)|\big) \right| = \left| h_i(x) - \Big(c_i \pm \big(r - d(y,u)\big)\Big) \right| \leqslant d(x,y)$$

follow from (2.3).

Set

$$f_i(y) = \sup_{x \in L} \big(f_i(x) + |g(x)| - d(x,y) \big) \quad \text{for every } y \in M \backslash L.$$

Note that, on $M \backslash L$, the function f_i agrees with a norm preserving extension of $(f_i + |g|)|_L$. It remains to show that $\|f_i \pm g\| \leqslant 1$. To this end, it suffices to show that, whenever $x, y \in M$, one has

$$-d(x,y) \leqslant f_i(x) \pm g(x) - \big(f_i(y) \pm g(y)\big) \leqslant d(x,y). \tag{2.4}$$

For the cases when $x, y \in L$ or $x, y \in M \backslash L$, or $x \in N$ (or $y \in N$) and $y \in M \backslash L$ (or $x \in M \backslash L$), the inequalities (2.4) follow from what has been proven above. So, in fact, it suffices to consider the case when $x \in B(u,r) \cup B(v,s)$ and $y \in M \backslash L$. In this case, (2.4) means that

$$-d(x,y) \leqslant c_i \pm g(x) - \sup_{z \in L} \big(f_i(z) + |g(z)| - d(z,y) \big) \leqslant d(x,y).$$

Thus, it suffices to show that

(1) there is a $z \in L$ such that

$$c_i \pm g(x) - d(x,y) + d(z,y) \leqslant f_i(z) + |g(z)|;$$

(2) for every $z \in L$,

$$f_i(z) + |g(z)| \leqslant c_i \pm g(x) + d(x,y) + d(z,y).$$

For (1), one may take $z = x$, so it remains to prove (2). By symmetry, it suffices to consider only the case when $x \in B(u,r)$. In this case $g(x) = r - d(x,u) \geqslant 0$. Thus, it suffices to prove that, for every $z \in L$,

$$f_i(z) + |g(z)| \leqslant c_i - r + d(x,u) + d(x,y) + d(z,y).$$

One has to look through the following cases:

(a) $z \in B(u,r)$; (b) $z \in B(v,s)$; (c) $z \in N$.

(a). If $z \in B(u,r)$, then $f_i(z) = c_i$ and $|g(z)| = r - d(z,u)$. Thus, one has to show that
$$2r \leqslant d(x,u) + d(z,u) + d(x,y) + d(z,y).$$
This inequality holds, because, since $y \notin B(u,r)$, one has $d(y,u) \geqslant r$, and thus
$$2r \leqslant d(y,u) + d(y,u) \leqslant d(x,u) + d(x,y) + d(z,u) + d(z,y).$$

(b). If $z \in B(v,s)$, then $f_i(z) = c_i$ and $|g(z)| = s - d(z,v)$. Thus, one has to show that
$$r + s \leqslant d(x,u) + d(z,v) + d(x,y) + d(z,y).$$
This inequality holds, because, since $y \notin B(u,r)$ and $y \notin B(v,s)$, one has $d(y,u) \geqslant r$ and $d(y,v) \geqslant s$, and thus
$$r + s \leqslant d(y,u) + d(y,v) \leqslant d(x,u) + d(x,y) + d(z,v) + d(z,y).$$

(c). If $z \in N$, then
$$\begin{aligned} f_i(z) + |g(z)| = f_i(z) = h_i(z) &\leqslant \breve{a}_i + d(z,u) \\ &\leqslant c_i - r + d(x,u) + d(x,y) + d(z,y). \end{aligned}$$
□

Proof of Theorem 2.3. (i)⇒(ii). Assume that $\mathrm{Lip}_0(M)$ has the w^*-SSD2P, let N be a finite subset of M and let $0 < \varepsilon < 1$. Choose $\alpha > 0$ such that $2\alpha < \varepsilon$ and, for all $x, y \in N$ with $x \neq y$,
$$\alpha \leqslant \frac{1}{d(x,y)} \quad \text{and} \quad 2\alpha \leqslant d(x,y).$$
For all $x, y \in N$ with $x \neq y$, define a weak*-slice $S_{x,y} = S\big(\frac{\delta_x - \delta_y}{d(x,y)}, \alpha^3\big)$ of $B_{\mathrm{Lip}_0(M)}$. Since $\mathrm{Lip}_0(M)$ has the w^*-SSD2P, we can find $f_{x,y} \in S_{x,y}$ and $g \in B_{\mathrm{Lip}_0(M)}$ with $\|g\| \geqslant 1 - \alpha$, such that $\|f_{x,y} \pm g\| \leqslant 1$. For $x, y \in N$ with $x = y$, define $f_{x,y} = 0 \in \mathrm{Lip}_0(M)$.

For all $x, y \in N$, one has
$$\langle f_{x,y}, \delta_x - \delta_y \rangle = f_{x,y}(x) - f_{x,y}(y) \geqslant (1 - \alpha^3) d(x,y),$$

therefore, keeping in mind that $\|f_{x,y} \pm g\| \leqslant 1$,

$$|\langle g, \delta_x - \delta_y\rangle| = |g(x) - g(y)| \leqslant \alpha^3 d(x,y) \leqslant \alpha^2.$$

Since $\|g\| \geqslant 1-\alpha$, there exist $u, v \in M$ with $u \neq v$, such that

$$\langle g, \delta_u - \delta_v\rangle = g(u) - g(v) \geqslant (1-\alpha)\, d(u,v).$$

Now, for any $x, y \in N$, again using that $\|f_{x,y} \pm g\| \leqslant 1$,

$$|\langle f_{x,y}, \delta_u - \delta_v\rangle| = |f_{x,y}(u) - f_{x,y}(v)| \leqslant \alpha d(u,v).$$

Letting $x, y, z, w \in N$ be arbitrary, it remains to verify (2.1) and (2.2). Since $\|f_{x,y} \pm g\| \leqslant 1$, we get

$$\begin{aligned}
&(1-\varepsilon)\big(d(u,v) + d(x,y)\big)\\
&\qquad \leqslant (1-2\alpha)d(u,v) + (1-2\alpha^3)d(x,y)\\
&\qquad \leqslant \langle g, \delta_u - \delta_v\rangle - \langle f_{x,y}, \delta_u - \delta_v\rangle + \langle f_{x,y}, \delta_x - \delta_y\rangle - \langle g, \delta_x - \delta_y\rangle\\
&\qquad = \langle g - f_{x,y}, \delta_u - \delta_x\rangle - \langle g - f_{x,y}, \delta_v - \delta_y\rangle\\
&\qquad \leqslant d(x,u) + d(y,v).
\end{aligned}$$

Thus, (2.1) holds. If $x = y$ and $z = w$, then (2.2) follows from (2.1) with y replaced by z. If $x \neq y$ or $z \neq w$, then

$$\alpha\big(d(x,y) + d(z,w)\big) \geqslant 2\alpha^2 \geqslant |\langle g, \delta_z - \delta_x + \delta_w - \delta_y\rangle|,$$

and thus, since $\|f_{x,y} \pm g\| \leqslant 1$,

$$\begin{aligned}
&(1-\varepsilon)\big(2d(u,v) + d(x,y) + d(z,w)\big)\\
&\qquad \leqslant 2\big(g(u) - g(v)\big) + (1-\alpha^3-\alpha)\big(d(x,y) + d(z,w)\big)\\
&\qquad \leqslant 2\langle g, \delta_u - \delta_v\rangle + \langle f_{x,y}, \delta_x - \delta_y\rangle + \langle f_{z,w}, \delta_z - \delta_w\rangle\\
&\qquad\qquad + \langle g, \delta_z - \delta_x + \delta_w - \delta_y\rangle\\
&\qquad = \langle g - f_{x,y}, \delta_u - \delta_x\rangle + \langle g + f_{x,y}, \delta_u - \delta_y\rangle\\
&\qquad\qquad - \langle g + f_{z,w}, \delta_v - \delta_z\rangle - \langle g - f_{z,w}, \delta_v - \delta_w\rangle\\
&\qquad \leqslant d(x,u) + d(y,u) + d(z,v) + d(w,v).
\end{aligned}$$

(ii)$\Rightarrow$(i). Assume that M has the SLTP. Let $n \in \mathbb{N}$, let $S_i = S(\mu_i, \alpha_i)$, $i = 1, \ldots, n$, be weak* slices of $B_{\mathrm{Lip}_0(M)}$, and let $0 < \varepsilon < 1$. It suffices to find $f_i \in S_i$, $i = 1, \ldots, n$, and $g \in B_{\mathrm{Lip}_0(M)}$ with $\|g\| \geqslant (1-\varepsilon)$ such that $f_i \pm g \in S_i$ for every $i \in \{1, \ldots, n\}$. We may assume that, for every $i \in \{1, \ldots, n\}$, one has $\mu_i = \sum_{j=1}^{n_i} \lambda_{ij}\delta_{x_{ij}}$ for some $n_i \in \mathbb{N}$, $\lambda_{ij} \in \mathbb{R}\backslash\{0\}$, and $x_{ij} \in M$, $j = 1, \ldots, n_i$.

Now $N \coloneqq \{0\} \cup \bigcup_{i=1}^{n}\{x_{i1}, \ldots, x_{in_i}\}$ is a finite subset of M. We may also assume that $2\varepsilon < \min_{1\leqslant i\leqslant n} \alpha_i$. This enables, for every $i \in \{1, \ldots, n\}$, to pick an $h_i \in S_i$ with $\|h_i\| \leqslant 1 - \varepsilon$.

By the SLTP, there exist $u, v \in M$ with $u \neq v$ satisfying (2.1) and (2.2) for all $x, y, z, w \in N$. By Lemma 2.4 there exist functions $f_1, \ldots, f_n, g \in \mathrm{Lip}_0(M)$ satisfying

- $f_i|_N = h_i|_N$ for every $i \in \{1, \ldots, n\}$;
- $g|_N = 0$ and $\|g\| \geqslant 1 - \varepsilon$;
- $\|f_i \pm g\| \leqslant 1$ for every $i \in \{1, \ldots, n\}$.

Fix such functions and notice that $f_i \in S_i$ and $f_i \pm g \in S_i$ for every $i \in \{1, \ldots, n\}$. Therefore, the space $\mathrm{Lip}_0(M)$ has the w^*-SSD2P. $\square$

2.2 Examples

We give two examples of metric spaces M that have the LTP but fail the SLTP. By [34, Theorem 3.1] and Theorem 2.3, this implies that the corresponding Lipschitz spaces $\mathrm{Lip}_0(M)$ have the w^*-SD2P but fail the w^*-SSD2P. We also show that, in the definition of the SLTP, the inequality (2.2) (for all $x, y, z, w \in N$) does not imply the inequality (2.1) (for all $x, y \in N$). We finish by showing that every infinite metric subspace of ℓ_1 has the SLTP.

EXAMPLE 2.5. Let $M = \{a_1, a_2, b_1, b_2\} \cup \{u_m, v_m \colon m \in \mathbb{N}\}$ be the metric space where, for all $i, j \in \{1, 2\}$ and all $m \in \mathbb{N}$,

$$d(a_i, b_j) = d(a_i, u_m) = d(b_i, v_m) = d(u_m, v_m) = 1,$$

and the distance between two different elements is 2 in all other cases (see Figure 2.1).

We first show that M has the LTP. Letting N be a finite subset of M and $m \in \mathbb{N}$ be such that $u_m, v_m \in M \backslash N$, it suffices to show that, for all $x, y \in N$, one has

$$d(x, y) + d(u_m, v_m) \leqslant d(x, u_m) + d(y, v_m).$$

If $d(x, u_m)+d(y, v_m) \geqslant 3$, then the desired inequality holds because $d(u_m, v_m) = 1$. If $d(x, u_m)+d(y, v_m) = 2$, then $x \in \{a_1, a_2\}$ and $y \in \{b_1, b_2\}$, and therefore $d(x, y) = 1$.

It remains to show that M fails the SLTP. Take $N = \{a_1, a_2, b_1, b_2\}$. Then, for all $u, v \in M$ with $u \neq v$, there exist $x, y, z, w \in N$ such that

$$2d(u, v) + d(x, y) + d(z, w) \geqslant d(x, u) + d(y, u) + d(z, v) + d(w, v) + 1.$$

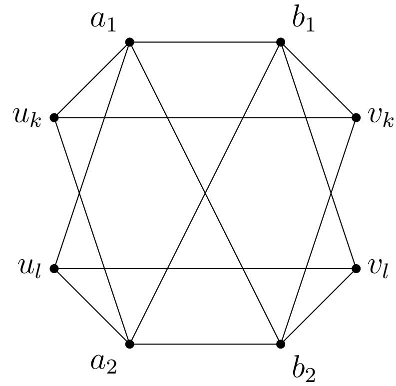

Figure 2.1: A representation of the metric space M in Example 2.5. The distances between points connected by a straight line segment are 1, the distances between other different points are 2.

Indeed, set $U = \{u_m : m \in \mathbb{N}\}$ and $V = \{v_m : m \in \mathbb{N}\}$, and suppose that $u, v \in M$ with $u \neq v$. If $u, v \in U$ or $u, v \in V$, then, respectively, for $x = z = a_1$, $y = w = a_2$, and for $x = z = b_1$, $y = w = b_2$,

$$\begin{aligned} 2d(u,v) + d(x,y) + d(z,w) &= 8 > 4 \\ &= d(x,u) + d(y,u) + d(z,v) + d(w,v). \end{aligned}$$

If $u \in U$ and $v \in V$, or $u \in V$ and $v \in U$, then, respectively, for $x = a_1$, $y = a_2$, $z = b_1$, $w = b_2$, and for $x = b_1$, $y = b_2$, $z = a_1$, $w = a_2$,

$$\begin{aligned} 2d(u,v) + d(x,y) + d(z,w) &\geqslant 6 > 4 \\ &= d(x,u) + d(y,u) + d(z,v) + d(w,v). \end{aligned}$$

Finally, if $u \in N$ or $v \in N$, then, respectively, for $x = y = u$ and $z, w \in N$ with $d(z,w) = 2$ and $d(z,v) = d(w,v) = 1$, and for $z = w = v$ and $x, y \in N$ with $d(x,y) = 2$ and $d(x,u) = d(y,u) = 1$,

$$\begin{aligned} 2d(u,v) + d(x,y) + d(z,w) &\geqslant 4 > 2 \\ &= d(x,u) + d(y,u) + d(z,v) + d(w,v). \end{aligned}$$

It is straightforward to verify that the metric space in the following example also has the LTP but not the SLTP.

Example 2.6. Let $M = \mathbb{N}$ be the metric space, where the distance between two integers with different parity is 1, and the distance between two distinct integers with the same parity is 2.

The following example shows that the inequality (2.2) in the definition of the SLTP does not imply (2.1).

EXAMPLE 2.7. Let $M = \{a, b\} \cup \{u_m, v_m : m \in \mathbb{N}\}$ be the metric space where, for all $m \in \mathbb{N}$,

$$d(a, u_m) = d(b, v_m) = d(u_m, v_m) = 1$$

and the distance between two different elements is 2 in all other cases (see Figure 2.2).

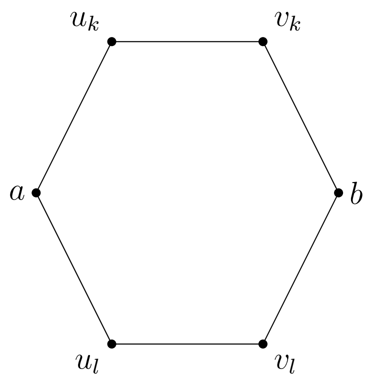

Figure 2.2: A representation of the metric space M in Example 2.7. The distances between points connected by a straight line segment are 1, the distances between other different points are 2.

For any finite subset N of M, we can find an $m \in \mathbb{N}$ such that $u_m, v_m \in M \backslash N$. We first show that, for all $x, y, z, w \in N$,

$$\begin{aligned} d(x,y) + d(z,w) + 2d(u_m, v_m) &= d(x,y) + d(z,w) + 2 \\ &\leqslant d(x,u_m) + d(y,u_m) + d(z,v_m) + d(w,v_m). \end{aligned}$$

By symmetry it suffices to show that, for all $x, y \in M \backslash \{u_m, v_m\}$,

$$d(x,y) + 1 \leqslant d(x,u_m) + d(y,u_m).$$

This inequality holds trivially if $d(x,u_m) + d(y,u_m) \geqslant 3$. It remains to note that if $d(x,u_m) + d(y,u_m) < 3$, then $d(x,u_m) = d(y,u_m) = 1$. Thus, $x = y = a$, and the desired inequality trivially holds.

We now show that M does not have the LTP. Take $N = \{a, b\}$. Then, for all $u, v \in M$ with $u \neq v$, there exist $x, y \in N$ such that

$$d(x,y) + d(u,v) \geqslant d(x,u) + d(y,v) + 1.$$

Indeed, set $U = \{u_m : m \in \mathbb{N}\}$ and $V = \{v_m : m \in \mathbb{N}\}$, and suppose that $u, v \in M$ with $u \neq v$. If $u, v \in U$ or $u, v \in V$, then, for $x = a$, $y = b$,

$$d(x,y) + d(u,v) = 4 > 3 = d(x,u) + d(y,v).$$

If $u \in U$ and $v \in V$, or $u \in V$ and $v \in U$, then, respectively, for $x = a$, $y = b$, and for $x = b$, $y = a$,

$$d(x,y) + d(u,v) \geqslant 3 > 2 = d(x,u) + d(y,v).$$

Finally, if $u \in N$ or $v \in N$, then, respectively, for $x = u$, $y \in N \backslash \{x\}$, and for $y = v$, $x \in N \backslash \{y\}$,

$$d(x,y) + d(u,v) \geqslant 3 > 2 \geqslant d(x,u) + d(y,v).$$

In [34, Proposition 4.7], it was shown that every infinite subset M of ℓ_1, viewed as a metric space, has the LTP. It turns out that every such M has even the SLTP.

Example 2.8. Every infinite subset M of ℓ_1, viewed as a metric space, has the SLTP.

Indeed, from [20, Theorem 5.6] combined with our Theorem 2.3 it follows that every unbounded metric space and every not uniformly discrete metric space M has the SLTP (this can also, without too much effort, be verified directly). Thus it suffices to consider the case when M is a bounded and uniformly discrete subset of ℓ_1. In this case there exist $R, r > 0$ such that, for all $x, y \in M$ with $x \neq y$, one has

$$r < d(x,y) < R.$$

Let N be a finite subset of M and let $\varepsilon > 0$. Choose $\delta > 0$ such that $\varepsilon r \geqslant 6\delta$. Since N is finite, there exists an $n \in \mathbb{N}$ such that for any $x = (x_m) \in N$

$$\sum_{m>n} |x_m| \leqslant \delta.$$

Since M is infinite and bounded, there exist $u = (u_m), v = (v_m) \in M$ with $u \neq v$, such that

$$\sum_{m \leqslant n} |u_m - v_m| \leqslant \delta.$$

For all $x = (x_m), y = (y_m) \in N$ and $a = (a_m), b = (b_m) \in \{u, v\}$,

$$\begin{aligned}\sum_{m} |x_m - y_m| &\leqslant \sum_{m \leqslant n} \big(|x_m - a_m| + |y_m - b_m| + |a_m - b_m|\big) + \sum_{m>n} |x_m - y_m| \\ &\leqslant \sum_{m \leqslant n} \big(|x_m - a_m| + |y_m - b_m|\big) + 3\delta\end{aligned}$$

and

$$\begin{aligned}\sum_m |u_m - v_m| &\leqslant \sum_{m>n} |u_m - v_m - x_m + y_m| + \sum_{m>n} |x_m - y_m| + \sum_{m\leqslant n} |u_m - v_m| \\ &\leqslant \sum_{m>n} \big(|x_m - u_m| + |y_m - v_m|\big) + 3\delta.\end{aligned}$$

Therefore, for all $x = (x_m), y = (y_m), z = (z_m), w = (w_m) \in N$,

$$\begin{aligned}(1-\varepsilon)(d(x,y) + d(u,v)) &\leqslant d(x,y) + d(u,v) - 6\delta \\ &= \sum_m |x_m - y_m| + \sum_m |u_m - v_m| - 6\delta \\ &\leqslant \sum_{m\leqslant n} \big(|x_m - u_m| + |y_m - v_m|\big) + 3\delta \\ &\qquad + \sum_{m>n} \big(|x_m - u_m| + |y_m - v_m|\big) + 3\delta - 6\delta \\ &= \sum_m \big(|x_m - u_m| + |y_m - v_m|\big) \\ &= d(x,u) + d(y,v)\end{aligned}$$

and

$$\begin{aligned}&(1-\varepsilon)\big(2d(u,v) + d(x,y) + d(z,w)\big) \\ &\quad \leqslant 2d(u,v) + d(x,y) + d(z,w) - 12\delta \\ &\quad = 2\sum_m |u_m - v_m| + \sum_m |x_m - y_m| + \sum_m |z_m - w_m| - 12\delta \\ &\quad \leqslant \sum_{m>n} \big(|x_m - u_m| + |z_m - v_m| + |y_m - u_m| + |w_m - v_m|\big) + 6\delta \\ &\qquad\quad + \sum_{m\leqslant n} \big(|x_m - u_m| + |y_m - u_m| + |z_m - v_m| + |w_m - v_m|\big) + 6\delta - 12\delta \\ &\quad = \sum_m \big(|x_m - u_m| + |y_m - u_m| + |z_m - v_m| + |w_m - v_m|\big) \\ &\quad = d(x,u) + d(y,u) + d(z,v) + d(w,v).\end{aligned}$$

Chapter 3

On the duality of the weak-star symmetric strong diameter two property in spaces of Lipschitz functions

We continue the study of the w^*-SSD2P for the spaces of Lipschitz functions. We characterise the w^*-SSD2P property in spaces of Lipschitz functions by a property of its predual, the Lipschitz-free space. We introduce the notion of decomposable octahedrality and study its duality with the symmetric strong diameter 2 property in general. This chapter is based on [32].

Let M be a pointed metric. Recall that the Lipschitz space $\mathrm{Lip}_0(M)$ has the w^*-SD2P if and only if the norm of the Lipschitz-free space $\mathcal{F}(M)$ is octahedral. In [34], it was shown that these properties can also be characterised via a property of the metric space M, namely the LTP.

Theorem 3.1 ([34, Theorem 3.1])**.** *Let M be a pointed metric space. The following statements are equivalent:*

(i) $\mathrm{Lip}_0(M)$ *has the w^*-SD2P;*

(ii) $\mathcal{F}(M)$ *is OH;*

(iii) *M has the LTP.*

In the previous chapter, we showed that the properties w^*-SD2P and w^*-SSD2P are different for the spaces of Lipschitz functions and gave a characterisation of the w^*-SSD2P for the space $\mathrm{Lip}_0(M)$ via a property of the metric space M, namely the SLTP. In this chapter, we give a characterisation of the latter properties in terms of the Lipschitz-free space $\mathcal{F}(M)$. To do so, we introduce the following property.

Definition 3.2. We say that a Banach space X is *decomposably octahedral* (briefly, *DOH*) if, whenever E is a finite subset of S_X and $\varepsilon > 0$, there exists a $y \in S_X$ such that, for all $y_1, \ldots, y_n \in X$ with $\sum_{i=1}^n y_i = y$, and, for all $x_1, \ldots, x_n \in E$ and $a_1, b_1, \ldots, a_n, b_n \geqslant 0$, one has

$$\sum_{i=1}^{n} \big(\|a_i x_i + y_i\| + \|b_i x_i - y_i\| \big) \geqslant (1-\varepsilon) \Big(\sum_{i=1}^{n} (a_i + b_i) + 2 \Big).$$

It is easy to verify that OH follows from DOH. We show that the Lipschitz space $\mathrm{Lip}_0(M)$ has the w^*-SSD2P if and only if $\mathcal{F}(M)$ is DOH. More generally, we show that a Banach space X is DOH whenever the dual space X^* has the w^*-SSD2P. We look through examples of octahedral Banach spaces whose duals are known not to have the w^*-SSD2P. All of these examples also fail the DOH.

3.1 Main results

We now show that if a dual Banach space X^* has the w^*-SSD2P, then X is DOH. Whether the reverse implication holds, in general, is unknown to us. However, in the following, we prove that the reverse implication holds if X is a Lipschitz-free space.

Proposition 3.3. *Let X be a Banach space. If X^* has the w^*-SSD2P, then X is DOH.*

Proof. Assume that X^* has the w^*-SSD2P. Let E be a finite subset of S_X and let $\varepsilon > 0$. For every $x \in E$, define a w^*-slice $S_x = S\big(x, \frac{\varepsilon}{2}\big)$ of B_{X^*}. Since X^* has the w^*-SSD2P, we can find $f_x \in S_x$ and $g \in B_{X^*}$ such that $\|f_x \pm g\| \leqslant 1$ for every $x \in E$ and $\|g\| \geqslant 1 - \varepsilon$. Then, for every $x \in E$,

$$f_x(x) \geqslant 1 - \frac{\varepsilon}{2} \quad \text{and} \quad |g(x)| \leqslant \frac{\varepsilon}{2}.$$

Let $y \in B_X$ be such that $g(y) \geqslant 1-\varepsilon$. For all $y_1, \ldots, y_n \in X$ with $\sum_{i=1}^n y_i = y$, and, for all $x_1, \dots, x_n \in E$ and $a_1, b_1, \dots, a_n, b_n \geqslant 0$, one has

$$\begin{aligned}
&\sum_{i=1}^{n}\big(\|a_i x_i + y_i\| + \|b_i x_i - y_i\|\big) \\
&\qquad \geqslant \sum_{i=1}^{n}\big((f_{x_i} + g)(a_i x_i + y_i) + (f_{x_i} - g)(b_i x_i - y_i)\big) \\
&\qquad = \sum_{i=1}^{n}\big((a_i + b_i) f_{x_i}(x_i) + (a_i - b_i) g(x_i) + 2g(y_i)\big) \\
&\qquad \geqslant (1-\varepsilon)\sum_{i=1}^{n}(a_i + b_i) + 2\sum_{i=1}^{n} g(y_i) \\
&\qquad = (1-\varepsilon)\sum_{i=1}^{n}(a_i + b_i) + 2g(y) \\
&\qquad \geqslant (1-\varepsilon)\Big(\sum_{i=1}^{n}(a_i + b_i) + 2\Big).
\end{aligned}$$

Therefore, X is DOH. □

Theorem 3.4. *Let M be a pointed metric space. The following statements are equivalent:*

(i) $\mathrm{Lip}_0(M)$ *has the w^*-SSD2P;*

(ii) $\mathcal{F}(M)$ *is DOH;*

(iii) M *has the SLTP.*

Proof. (i)⇔(iii) is Theorem 2.3.

(i)⇒(ii) holds by Proposition 3.3.

(ii)⇒(iii). Assume that the Lipschitz-free space $\mathcal{F}(M)$ is DOH. Let N be a finite subset of M and let $0 < \varepsilon < \frac{1}{2}$. Define $E = \big\{\frac{\delta_p - \delta_q}{d(p,q)} : p, q \in N\big\}$ and let $0 < \delta < \frac{r\varepsilon}{2R}$, where $r, R > 0$ are such that, for all $p, q \in N$ with $p \neq q$, one has

$$r < d(p,q) < R.$$

Since $\mathcal{F}(M)$ is DOH, there exists a $\nu \in \operatorname{span}\{\delta_p - \delta_q \colon p, q \in M\}$ with $\|\nu\| < 1$, such that, for all $\nu_1, \dots, \nu_n \in \mathcal{F}(M)$ with $\sum_{i=1}^{n} \nu_i = \nu$, and, for all $\mu_1, \dots, \mu_n \in E$ and $a_1, b_1, \dots, a_n, b_n \geqslant 0$, one has

$$\sum_{i=1}^{n}\big(\|a_i \mu_i + \nu_i\| + \|b_i \mu_i - \nu_i\|\big) \geqslant (1-\delta)\Big(\sum_{i=1}^{n}(a_i + b_i) + 2\Big).$$

Since

$$\|\nu\| = \inf\Big\{\sum_{i=1}^{n} |\lambda_i| d(p_i, q_i) \colon \nu = \sum_{i=1}^{n} \lambda_i(\delta_{p_i} - \delta_{q_i}),\ p_i, q_i \in M\Big\},$$

there exist $n \in \mathbb{N}$, $\lambda_1, \dots, \lambda_n > 0$, and $u_1, v_1, \dots, u_n, v_n \in M$ with $u_i \neq v_i$ for every $i \in \{1, \dots, n\}$, such that $\nu = \sum_{i=1}^{n} \lambda_i(\delta_{u_i} - \delta_{v_i})$ and $\sum_{i=1}^{n} \lambda_i d(u_i, v_i) = 1$.

It suffices to show that there exists an $i \in \{1, \dots, n\}$ such that, taking $u = u_i$ and $v = v_i$, one has

$$(1-\varepsilon)\big(d(x,y) + d(u,v)\big) \leqslant d(x,u) + d(y,v)$$

for all $x, y \in N$, and

$$\begin{aligned}(1-\varepsilon)\big(2d(u,v) + d(x,y) + d(z,w)\big)&\\ \leqslant d(x,u) + d(y,u) + d(z,v) + d(w,v)&\end{aligned}$$

for all $x, y, z, w \in N$. Suppose that, contrary to our claim, for every $i \in \{1, \dots, n\}$, there exist $x_i, y_i \in N$ such that

$$(1-\varepsilon)\big(d(x_i,y_i) + d(u_i,v_i)\big) > d(x_i,u_i) + d(y_i,v_i), \tag{3.1}$$

or there exist $x_i', y_i', z_i, w_i \in N$ such that

$$\begin{aligned}(1-\varepsilon)\big(d(x_i',y_i') + d(z_i,w_i) + 2d(u_i,v_i)\big)&\\ > d(x_i',u_i) + d(y_i',u_i) + d(z_i,v_i) + d(w_i,v_i).&\end{aligned} \tag{3.2}$$

Let I be the subset of indexes $\{1, \dots, n\}$ for which there exist $x_i, y_i \in N$ such that (3.1) holds, and let J be the set $\{1, \dots, n\} \backslash I$. By our assumption, for every $i \in J$, there exist $x_i', y_i', z_i, w_i \in N$ such that $x_i' \neq y_i'$ or $z_i \neq w_i$, and (3.2) holds. Fix such $x_i, y_i \in N$ for every $i \in I$, and x_i', y_i', z_i, w_i for every $i \in J$. Then

$$\begin{aligned}\sum_{i\in I} \lambda_i\Big(d(x_i,u_i) + d(y_i,v_i) - (1-\varepsilon)\big(d(x_i,y_i) + d(u_i,v_i)\big)\Big)&\\ + \sum_{i\in J} \lambda_i\big(d(x_i',u_i) + d(y_i',u_i) + d(z_i,v_i) + d(w_i,v_i)\big)&\\ - (1-\varepsilon)\sum_{i\in J} \lambda_i\big(d(x_i',y_i') + d(z_i,w_i) + 2d(u_i,v_i)\big) < 0.&\end{aligned}$$

To prove that this cannot be the case, we show that the following inequality holds,

$$\begin{aligned}
&\sum_{i\in I}\lambda_i\Big(d(x_i,u_i)+d(y_i,v_i)-(1-\varepsilon)\big(d(x_i,y_i)+d(u_i,v_i)\big)\Big)\\
&\qquad\qquad+\sum_{i\in J}\lambda_i\big(d(x_i',u_i)+d(y_i',u_i)+d(z_i,v_i)+d(w_i,v_i)\big)\\
&\qquad\qquad-(1-\varepsilon)\sum_{i\in J}\lambda_i\big(d(x_i',y_i')+d(z_i,w_i)+2d(u_i,v_i)\big)\\
&\qquad\geqslant\sum_{i\in I}\lambda_i\big(d(x_i,u_i)+d(y_i,v_i)+d(u_i,v_i)\big)\\
&\qquad\qquad-(1-\delta)\sum_{i\in I}\lambda_i\big(d(x_i,y_i)+2d(u_i,v_i)\big)\\
&\qquad\qquad+\sum_{i\in J}\lambda_i\big(d(x_i',u_i)+d(y_i',u_i)+d(z_i,v_i)+d(w_i,v_i)+\delta d(y_i',z_i)\big)\\
&\qquad\qquad-(1-\delta)\sum_{i\in J}\lambda_i\big(d(x_i',y_i')+d(z_i,w_i)+2d(u_i,v_i)\big),
\end{aligned}$$

and that the right hand side of this inequality is nonnegative.

The inequality holds because, since $2\delta\leqslant\varepsilon$, for every $i\in I$, one has

$$\begin{aligned}
&d(x_i,u_i)+d(y_i,v_i)-(1-\varepsilon)\big(d(x_i,y_i)+d(u_i,v_i)\big)\\
&\qquad\geqslant d(x_i,u_i)+d(y_i,v_i)-(1-2\delta)\big(d(x_i,y_i)+d(u_i,v_i)\big)\\
&\qquad\geqslant d(x_i,u_i)+d(y_i,v_i)+d(u_i,v_i)-(1-\delta)\big(d(x_i,y_i)+2d(u_i,v_i)\big),
\end{aligned}$$

and, for every $i\in J$, since $\delta d(y_i',z_i)\leqslant\frac{\varepsilon}{2}\big(d(x_i',y_i')+d(z_i,w_i)\big)$, one has

$$\begin{aligned}
&d(x_i',u_i)+d(y_i',u_i)+d(z_i,v_i)+d(w_i,v_i)\\
&\qquad\qquad-(1-\varepsilon)\big(d(x_i',y_i')+d(z_i,w_i)+2d(u_i,v_i)\big)\\
&\qquad\geqslant d(x_i',u_i)+d(y_i',u_i)+d(z_i,v_i)+d(w_i,v_i)\\
&\qquad\qquad-(1-\delta-\frac{\varepsilon}{2})\big(d(x_i',y_i')+d(z_i,w_i)+2d(u_i,v_i)\big)\\
&\qquad\geqslant d(x_i',u_i)+d(y_i',u_i)+d(z_i,v_i)+d(w_i,v_i)+\delta d(y_i',z_i)\\
&\qquad\qquad-(1-\delta)\big(d(x_i',y_i')+d(z_i,w_i)+2d(u_i,v_i)\big).
\end{aligned}$$

It remains to prove that

$$\sum_{i\in I}\lambda_i\Big(d(x_i,u_i)+d(y_i,v_i)+d(u_i,v_i)-(1-\delta)\big(d(x_i,y_i)+2d(u_i,v_i)\big)\Big)$$
$$+\sum_{i\in J}\lambda_i\big(d(x_i',u_i)+d(y_i',u_i)+d(z_i,v_i)+d(w_i,v_i)+\delta d(y_i',z_i)\big)$$
$$-(1-\delta)\sum_{i\in J}\lambda_i\big(d(x_i',y_i')+d(z_i,w_i)+2d(u_i,v_i)\big)\geqslant 0.$$

To this end, note that

$$\begin{aligned}
&\sum_{i\in I}\lambda_i\big(d(x_i,u_i)+d(y_i,v_i)+d(u_i,v_i)\big)\\
&\qquad+\sum_{i\in J}\lambda_i\big(d(x_i',u_i)+d(y_i',u_i)+d(y_i',z_i)+d(z_i,v_i)+d(w_i,v_i)\big)\\
&\geqslant\sum_{i\in I}\lambda_i\big(\|\delta_{x_i}-\delta_{y_i}-(\delta_{u_i}-\delta_{v_i})\|+\|\delta_{u_i}-\delta_{v_i}\|\big)\\
&\qquad+\sum_{i\in J}\lambda_i\big(\|\delta_{x_i'}-\delta_{y_i'}-(\delta_{u_i}-\delta_{y_i'})\|+\|\delta_{u_i}-\delta_{y_i'}\|\big)\\
&\qquad+\sum_{i\in J}\lambda_i\big(\|\delta_{y_i'}-\delta_{z_i}-(\delta_{y_i'}-\delta_{z_i})\|+\|\delta_{y_i'}-\delta_{z_i}\|\big)\\
&\qquad+\sum_{i\in J}\lambda_i\big(\|\delta_{z_i}-\delta_{w_i}-(\delta_{z_i}-\delta_{v_i})\|+\|\delta_{z_i}-\delta_{v_i}\|\big)\\
&\geqslant 2(1-\delta)+(1-\delta)\sum_{i\in I}\lambda_i d(x_i,y_i)\\
&\qquad+(1-\delta)\sum_{i\in J}\lambda_i\big(d(x_i',y_i')+d(y_i',z_i)+d(z_i,w_i)\big)\\
&=(1-\delta)\sum_{i\in I}\lambda_i\big(d(x_i,y_i)+2d(u_i,v_i)\big)\\
&\qquad+(1-\delta)\sum_{i\in J}\lambda_i\big(d(x_i',y_i')+d(y_i',z_i)+d(z_i,w_i)+2d(u_i,v_i)\big),
\end{aligned}$$

where the second inequality holds by our choice of ν because

$$\nu=\sum_{i\in I}(\delta_{u_i}-\delta_{v_i})+\sum_{i\in J}(\delta_{u_i}-\delta_{y_i'}+\delta_{y_i'}-\delta_{z_i}+\delta_{z_i}-\delta_{v_i}).$$

This completes the proof. □

3.2 Decomposable octahedrality in Banach spaces

In this section, we look at examples of octahedral Banach spaces for which it is known that the dual space does not have the w^*-SSD2P. These Banach spaces also fail to be decomposably octahedral. This leaves open the question of whether the reverse implication of Proposition 3.3 holds.

We start by looking at decomposable octahedrality in absolute sums of Banach spaces. Recall that a norm N on $\mathbb{R}^2$ is *absolute* if

$$N(a,b) = N(|a|,|b|) \qquad \text{for all } (a,b) \in \mathbb{R}^2,$$

and *normalised* if

$$N(1,0) = N(0,1) = 1.$$

If N is an absolute normalised norm on $\mathbb{R}^2$ (see [14, Lemmata 21.1 and 21.2]), then

(a) $\|(a,b)\|_\infty \leqslant N(a,b) \leqslant \|(a,b)\|_1$ for all $(a,b) \in \mathbb{R}^2$;

(b) if $(a,b),(c,d) \in \mathbb{R}^2$ with $|a| \leqslant |c|$ and $|b| \leqslant |d|$, then

$$N(a,b) \leqslant N(c,d);$$

(c) the dual norm N^* on $\mathbb{R}^2$ defined by

$$N^*(c,d) = \max_{N(a,b)\leqslant 1} (|ac| + |bd|) \qquad \text{for all } (c,d) \in \mathbb{R}^2$$

is also absolute and normalised. Note that $(N^*)^* = N$.

For $1 \leqslant p \leqslant \infty$, we denote the ℓ_p norm on $\mathbb{R}^2$ by $\|\cdot\|_p$. Every ℓ_p norm is an absolute normalised norm.

For Banach spaces X and Y, we denote by $X \oplus_N Y$ the product space $X \times Y$ equipped with the norm N, where

$$N(x,y) = N(\|x\|,\|y\|) \qquad \text{for all } x \in X, y \in Y.$$

In case N is an ℓ_p norm we write $X \oplus_p Y$. Note that $(X \oplus_N Y)^* = X^* \oplus_{N^*} Y^*$.

It can be shown that $X^* \oplus_N Y^*$ has the w^*-SSD2P if and only if N is the ℓ_∞ norm and X^* or Y^* has the w^*-SSD2P (the proof is similar to the one of [20, Theorem 3.1]).

We give necessary and sufficient conditions for the absolute sum of Banach spaces to be DOH. Note that Proposition 3.5 (a) does not extend to infinite sums because, by [20, Proposition 3.4] and Proposition 3.3, the space $\ell_1(X_n)$ is DOH for all sequences of Banach spaces $(X_n)_{n=1}^\infty$.

Proposition 3.5. *Let X and Y be Banach spaces.*

(a) *The space $X \oplus_1 Y$ is DOH if and only if X or Y is DOH.*

(b) *If N is an absolute normalised norm different from the ℓ_1 norm, then the space $X \oplus_N Y$ is not DOH.*

Proof. (a). First, assume that X is DOH. Let E be a finite subset of $S_{X\oplus_1 Y}$ and let $\varepsilon > 0$. Since X is DOH, there exists a $z \in S_X$ such that, for all $z_1, \ldots, z_n \in X$ with $\sum_{i=1}^n z_i = z$, and, for all $a_1, b_1, \ldots, a_n, b_n \geqslant 0$ and $(x_1, y_1), \ldots, (x_n, y_n) \in E$, one has

$$\sum_{i=1}^{n}\big(\|a_i x_i + z_i\| + \|b_i x_i - z_i\|\big) \geqslant (1-\varepsilon)\Big(\sum_{i=1}^{n}(a_i+b_i)\|x_i\| + 2\Big).$$

Notice that $N(z, 0) = 1$. Let $(z_1, w_1), \ldots, (z_n, w_n) \in X \oplus_1 Y$ be such that $\sum_{i=1}^n (z_i, w_i) = (z, 0)$. Then, for all $a_1, b_1, \ldots, a_n, b_n \geqslant 0$ and $(x_1, y_1), \ldots,$ $(x_n, y_n) \in E$, one has

$$\begin{aligned}
&\sum_{i=1}^{n}\big(\|a_i(x_i,y_i) + (z_i,w_i)\| + \|b_i(x_i,y_i) - (z_i,w_i)\|\big)\\
&\qquad = \sum_{i=1}^{n}\big(\|a_i x_i + z_i\| + \|a_i y_i + w_i\| + \|b_i x_i - z_i\| + \|b_i y_i - w_i\|\big)\\
&\qquad \geqslant (1-\varepsilon)\Big(\sum_{i=1}^{n}(a_i+b_i)\|x_i\| + 2\Big) + \sum_{i=1}^{n}(a_i+b_i)\|y_i\|\\
&\qquad \geqslant (1-\varepsilon)\Big(\sum_{i=1}^{n}(a_i+b_i) + 2\Big).
\end{aligned}$$

Therefore, $X \oplus_1 Y$ is DOH.

Assume now that X and Y are not DOH. Then there exist finite subsets E_1, E_2 of S_X and S_Y, respectively, and $\varepsilon > 0$, such that, for all $z \in S_X$, $w \in S_Y$, there exist $n \in \mathbb{N}$ and $z_i \in X$, $w_i \in Y$, $a_i, b_i, c_i, d_i \geqslant 0$, $x_i \in E_1$, $y_i \in E_2$, $i = 1, \ldots, n$, such that $\sum_{i=1}^n z_i = z$, $\sum_{i=1}^n w_i = w$,

$$\sum_{i=1}^{n}\big(\|a_i x_i + z_i\| + \|b_i x_i - z_i\|\big) < (1-\varepsilon)\Big(\sum_{i=1}^{n}(a_i+b_i) + 2\Big),$$

and

$$\sum_{i=1}^{n}\big(\|c_i y_i + w_i\| + \|d_i y_i - w_i\|\big) < (1-\varepsilon)\Big(\sum_{i=1}^{n}(c_i+d_i) + 2\Big).$$

Now, take $E = \{(x,0),(0,y)\colon x \in E_1, y \in E_2\}$. This is a finite subset of $S_{X\oplus_1 Y}$. However, for all $(\overline{z},\overline{w}) \in S_{X\oplus_1 Y}$, there exist $n \in \mathbb{N}$ and $\overline{z}_i \in X$, $\overline{w}_i \in Y$, $a_i, b_i, c_i, d_i \geqslant 0$, $(x_i,0),(0,y_i) \in E$, $i = 1,\ldots,n$, such that $\sum_{i=1}^n \overline{z}_i = \overline{z}$, $\sum_{i=1}^n \overline{w}_i = \overline{w}$, and

$$
\begin{aligned}
&\sum_{i=1}^n \big(\big\|\|\overline{z}\|a_i(x_i,0) + (\overline{z}_i,0)\big\| + \big\|\|\overline{z}\|b_i(x_i,0) - (\overline{z}_i,0)\big\|\big)\\
&\qquad\qquad + \sum_{i=1}^n \big(\big\|\|\overline{w}\|c_i(0,y_i) + (0,\overline{w}_i)\big\| + \big\|\|\overline{w}\|d_i(0,y_i) - (0,\overline{w}_i)\big\|\big)\\
&\quad = \sum_{i=1}^n \big(\big\|\|\overline{z}\|a_ix_i + \overline{z}_i\big\| + \big\|\|\overline{z}\|b_ix_i - \overline{z}_i\big\|\big)\\
&\qquad\qquad + \sum_{i=1}^n \big(\big\|\|\overline{w}\|c_iy_i + \overline{w}_i\big\| + \big\|\|\overline{w}\|d_iy_i - \overline{w}_i\big\|\big)\\
&\quad < (1-\varepsilon)\big(\|\overline{z}\|(a_i+b_i) + \|\overline{w}\|(c_i+d_i) + 2\big).
\end{aligned}
$$

Therefore, $X \oplus_1 Y$ is not DOH.

(b). Take $x \in S_X$ and $w \in S_Y$. Then $(x,0),(0,w) \in S_{X\oplus_N Y}$. Notice that, for every $y \in S_{X\oplus_N Y}$, there exist $y_1 \in B_X$, $y_2 \in B_Y$ such that $y = (y_1,0) + (0,y_2)$. Since N is not the ℓ_1 norm, there exists an $\varepsilon > 0$ such that $N(1,1) < 2(1-2\varepsilon)$. Thus

$$
\begin{aligned}
&\big\|\|y_1\|(x,0) + (0,y_2)\big\| + \big\|\|y_2\|(x,0) - (0,y_2)\big\|\\
&\qquad + \big\|\|y_1\|(0,w) + (y_1,0)\big\| + \big\|\|y_2\|(0,w) - (y_1,0)\big\|\\
&\quad = N(\|y_1\|,\|y_2\|) + N(\|y_2\|,\|y_2\|) + N(\|y_1\|,\|y_1\|) + N(\|y_2\|,\|y_1\|)\\
&\quad = 2 + \|y_2\|N(1,1) + \|y_1\|N(1,1)\\
&\quad < 2 + 2(1-2\varepsilon)\big(\|y_1\| + \|y_2\|\big)\\
&\quad \leqslant (1-\varepsilon)\big(2\|y_1\| + 2\|y_2\| + 2\big).
\end{aligned}
$$

Therefore, $X \oplus_N Y$ is not DOH. □

Note that, by Proposition 3.5, the space $\ell_1 \oplus_\infty \ell_1$, which is known to be OH (see [21]), is not DOH.

Let K be a compact Hausdorff space. It is known and straightforward to prove that the space $C(K)$ of all continuous functions on K is OH if and only if K does not have isolated points.

We finish the chapter by noting that the space $C(K)$ is not DOH. It suffices to find $f_1, f_2 \in S_{C(K)}$ such that, for every $g \in S_{C(K)}$, there exist $g_1, g_2 \in C(K)$ with $g = g_1 + g_2$ and

$$
\|f_1 + g_1\| + \|f_1 - g_1\| + \|f_2 + g_2\| + \|f_2 - g_2\| \leqslant 4.
$$

Fix $x, y \in K$ with $x \neq y$, and let G_x and G_y be disjoint closed neighbourhoods of x and y, respectively. By Urysohn's lemma, there exist continuous functions $f_1, f_2 \colon K \to [0,1]$ such that

$$f_1(z) = \begin{cases} 1 & \text{if } z = x; \\ 0 & \text{if } z \notin G_x, \end{cases} \qquad f_2(z) = \begin{cases} 1 & \text{if } z = y; \\ 0 & \text{if } z \notin G_y, \end{cases}$$

and a continuous function $h \colon K \to [0,1]$ such that

$$h(z) = \begin{cases} 0 & \text{if } z \in G_x; \\ 1 & \text{if } z \in G_y. \end{cases}$$

For every $g \in S_{C(K)}$, define $g_1 \colon K \to \mathbb{R}$ by $g_1(z) = g(z)h(z)$, and set $g_2 = g - g_1$. Then $g = g_1 + g_2$ and $f_1 \pm g_1$, $f_2 \pm g_2 \in S_{C(K)}$. Therefore,

$$\|f_1 + g_1\| + \|f_1 - g_1\| + \|f_2 + g_2\| + \|f_2 - g_2\| = 4.$$

Chapter 4

Diameter two properties in spaces of Lipschitz functions

In this chapter, we study diameter 2 properties in the spaces of Lipschitz functions and solve some open problems. Namely, we show that: the D2P, the SD2P, and the SSD2P are three different properties for these spaces of Lipschitz functions; the space $\mathrm{Lip}_0(K_n)$ has the SSD2P for every $n \in \mathbb{N}$, including the case of $n = 2$; every local norm-one Lipschitz function is a Daugavet-point. This chapter is based on [22].

Recall that the Banach space X has the

- *slice diameter* 2 *property* (briefly, *slice-D2P*) if every slice of B_X has diameter 2;
- *diameter* 2 *property* (briefly, *D2P*) if every nonempty relatively weakly open subset of B_X has diameter 2;
- *strong diameter* 2 *property* (briefly, *SD2P*) if every convex combination of slices of B_X has diameter 2, i.e., the diameter of $\sum_{i=1}^{n} \lambda_i S_i$ is 2 whenever $n \in \mathbb{N}$, $\lambda_1, \ldots, \lambda_n \geqslant 0$ with $\sum_{i=1}^{n} \lambda_i = 1$, and $S_1, \ldots, S_n$ are slices of B_X;
- *symmetric strong diameter* 2 *property* (briefly, *SSD2P*) if, for every $n \in \mathbb{N}$, every family $\{S_1, \ldots, S_n\}$ of slices of B_X, and every $\varepsilon > 0$, there exist $f_1 \in S_1, \ldots, f_n \in S_n$, and $g \in B_X$ with $\|g\| > 1 - \varepsilon$ such that $f_i \pm g \in S_i$ for every $i \in \{1, \ldots, n\}$.

If X is a dual space, then we also consider the weak* versions of these diameter 2 properties (w^*-slice-D2P, w^*-D2P, w^*-SD2P, and w^*-SSD2P), where slices and weakly open subsets in the above definitions are replaced by weak* slices and weak* open subsets, respectively.

In this chapter we continue studying diameter 2 properties in the space $\mathrm{Lip}_0(M)$. Diameter 2 properties of $\mathrm{Lip}_0(M)$ have been studied in [23], [34], [20], [33], [10], and [28]. In [23], Ivakhno proved that if a metric space M is unbounded or not uniformly discrete, then the space $\mathrm{Lip}_0(M)$ has the slice-D2P. In [10], Cascales et al. proved that if a metric space M has infinitely many cluster points or M is discrete but not uniformly discrete, then the space $\mathrm{Lip}_0(M)$ has even the SSD2P. In [28], Langemets and Rueda Zoca generalised this result by proving that the same is true if M is unbounded or not uniformly discrete.

Theorem 4.1 (see [28, Theorem 2.2]). *If the metric space M is unbounded or not uniformly discrete, then the space $\mathrm{Lip}_0(M)$ has the SSD2P.*

Theorem 4.1 leaves open for which bounded but not uniformly discrete metric spaces M the space $\mathrm{Lip}_0(M)$ has the SSD2P (or SD2P or D2P or slice-D2P). In [28], Langemets and Rueda Zoca proved that the space $\mathrm{Lip}_0(K_n)$ has the SSD2P whenever $n \in \mathbb{N}\backslash\{2\}$. Recall that K_n is the metric subspace of ℓ_∞ where the terms of the sequences are from the set $\{0, 1, \ldots, n\}$. In this chapter (see Theorem 4.7 and Proposition 4.10 below), we show that, in fact, this is true for *every* $n \in \mathbb{N}$, including the case of $n = 2$.

Theorem 4.2 (cf. [28, Propositions 2.7 and 2.8]). *For every $n \in \mathbb{N}$, the space $\mathrm{Lip}_0(K_n)$ has the SSD2P.*

In most cases, it seems to be unknown whether for $\mathrm{Lip}_0(M)$ the above-mentioned diameter 2 properties differ from each other. For example, in [28, Introduction], the authors say that it is not known whether the slice-D2P implies the SSD2P within the class of spaces of Lipschitz functions, and it is not known whether the SSD2P and the w^*-SSD2P coincide in general. Our Example 4.17, combined with Theorem 4.15, shows that the SD2P does not follow from w^*-SSD2P for the spaces of Lipschitz functions. Our Example 4.19, combined with Lemma 4.18, shows that the D2P does not follow from the w^*-SD2P for the spaces of Lipschitz functions. Therefore, we have the following result.

Theorem 4.3. *The SSD2P, the SD2P, and the D2P are three different properties for the spaces of Lipschitz functions. In fact, the w^*-SSD2P and the SD2P are different, and the w^*-SD2P and the D2P are different for the spaces of Lipschitz functions.*

It remains open whether, for the spaces of Lipschitz functions, any of the above-mentioned (non-weak*) diameter 2 properties coincides with its weak* version. In fact, we do not even know if, for the spaces of Lipschitz functions, the w^*-SSD2P implies the slice-D2P. It also remains open whether there exists a metric space M such that $\mathrm{Lip}_0(M)$ has the slice-D2P but not the D2P.

In [25], Jung and Rueda Zoca studied Daugavet-points in $\mathrm{Lip}_0(M)$ in connection with locality properties of Lipschitz functions. They posed and addressed the question of whether every local norm-one f in $\mathrm{Lip}_0(M)$ is a Daugavet-point. We answer that question with the following theorem.

Theorem 4.4 (cf. [25, Proposition 3.4 and Theorem 3.6])**.** *Let M be a pointed metric space. If $f \in S_{\mathrm{Lip}_0(M)}$ is local, then f is a Daugavet-point.*

In general, one faces difficulties when dealing with the dual space $\mathrm{Lip}_0(M)^*$ due to the lack of a useful characterisation of this space. Our results will heavily rely on the following observation. Set

$$\widetilde{M} = (M \times M) \backslash \{(x, x) \colon x \in M\}. \tag{4.1}$$

Then the mapping, also called the *de Leeuw's transform* (see, e.g., [36]),

$$\mathrm{Lip}_0(M) \ni f \longmapsto \widetilde{f} \in \ell_\infty(\widetilde{M}) \quad \text{where} \quad \widetilde{f}(x, y) = \frac{f(x) - f(y)}{d(x, y)},$$

is linear and isometric. Recall that the dual space of $\ell_\infty(\widetilde{M})$ is canonically isometrically isomorphic to the Banach space $ba(\widetilde{M})$ of all bounded and finitely additive signed measures μ on $\widetilde{M}$ with the total variation as the norm, that is $\|\mu\| = |\mu|(\widetilde{M})$. Thus, whenever $F \in \mathrm{Lip}_0(M)^*$, by the Hahn–Banach extension theorem, there is a $\mu \in ba(\widetilde{M})$ such that $|\mu|(\widetilde{M}) = \|F\|$ and

$$F(f) = \int_{\widetilde{M}} \widetilde{f} \, d\mu \quad \text{for every } f \in \mathrm{Lip}_0(M). \tag{4.2}$$

We are going to use the following notation. For a subset A of M, we write

$$\Gamma_{1,A} = \{(x, y) \in \widetilde{M} \colon x \in A\} \quad \text{and} \quad \Gamma_{2,A} = \{(x, y) \in \widetilde{M} \colon y \in A\}.$$

4.1 The SSD2P in spaces of Lipschitz functions

We start this section by giving two sufficient conditions for the space $\mathrm{Lip}_0(M)$ to have the SSD2P. The first one is a consequence of identifying the dual space of $\mathrm{Lip}_0(M)$ via the de Leeuw's transform. From this, we derive the second one, which involves only conditions on the metric of the space M and which we will then use to prove Theorems 4.1 and 4.2.

Lemma 4.5. *Let M be a pointed metric space and let $\widetilde{M}$ be as in* (4.1). *Suppose that, whenever $\delta > 0$, $n \in \mathbb{N}$, $h_1, \ldots, h_n \in \mathrm{Lip}_0(M)$ with $\|h_i\| \leqslant 1-\delta$ for every $i \in \{1, \ldots, n\}$, and $\mu \in ba(\widetilde{M})$ with only non-negative values, there exist a subset A of M and functions $f_1, \ldots, f_n, g \in \mathrm{Lip}_0(M)$ satisfying*

- $\mu(\Gamma_{1,A}) < \delta$ *and* $\mu(\Gamma_{2,A}) < \delta$;
- $f_i|_{M\setminus A} = h_i|_{M\setminus A}$ *for every* $i \in \{1, \ldots, n\}$;
- $g|_{M\setminus A} = 0$ *and* $\|g\| \geqslant 1-\delta$;
- $\|f_i \pm g\| \leqslant 1$ *for every* $i \in \{1, \ldots, n\}$.

Then the space $\mathrm{Lip}_0(M)$ has the SSD2P.

Proof. Let $n \in \mathbb{N}$, let $F_1, \ldots, F_n \in S_{\mathrm{Lip}_0(M)^*}$, and let $\varepsilon > 0$. It suffices to find $f_i \in S(F_i, \varepsilon)$, $i = 1, \ldots, n$, and $g \in \mathrm{Lip}_0(M)$ with $\|g\| > 1-\varepsilon$ such that $f_i \pm g \in S(F_i, \varepsilon)$ for every $i \in \{1, \ldots, n\}$.

For every $i \in \{1, \ldots, n\}$, let $\mu_i \in ba(\widetilde{M})$ with $|\mu_i|(\widetilde{M}) = 1$ satisfy (4.2) with F and μ replaced by F_i and μ_i, respectively. Define $\mu = |\mu_1| + \cdots + |\mu_n|$. Fix a real number $\delta > 0$ satisfying $8\delta \leqslant \varepsilon$. For every $i \in \{1, \ldots, n\}$, pick a function $h_i \in S(F_i, 2\delta)$ with $\|h_i\| \leqslant 1-\delta$.

Let a subset A of M and functions $f_1, \ldots, f_n, g \in \mathrm{Lip}_0(M)$ satisfy the conditions in the lemma. Setting $\Gamma_A = \Gamma_{1,A} \cup \Gamma_{2,A}$, one has $\mu(\Gamma_A) < 2\delta$, hence, whenever $i \in \{1, \ldots, n\}$,

$$|F_i(g)| = \left| \int_{\widetilde{M}} \widetilde{g}\, d\mu_i \right| = \left| \int_{\Gamma_A} \widetilde{g}\, d\mu_i \right| \leqslant |\mu_i|(\Gamma_A) < 2\delta$$

and (observing that $\widetilde{f_i}|_{\widetilde{M}\setminus\Gamma_A} = \widetilde{h}_i|_{\widetilde{M}\setminus\Gamma_A}$)

$$\begin{aligned} F_i(f_i) &= \int_{\widetilde{M}} \widetilde{f_i}\, d\mu_i = \int_{\widetilde{M}} \widetilde{h}_i\, d\mu_i + \int_{\Gamma_A} \big(\widetilde{f_i} - \widetilde{h}_i\big)\, d\mu_i \\ &= F_i(h_i) + \int_{\Gamma_A} \big(\widetilde{f_i} - \widetilde{h}_i\big)\, d\mu_i \\ &> 1 - 2\delta - 2|\mu_i|(\Gamma_A) > 1 - 6\delta, \end{aligned}$$

and thus

$$F_i(f_i \pm g) \geqslant F_i(f_i) - |F_i(g)| > 1 - 6\delta - 2\delta \geqslant 1 - \varepsilon.$$

$\square$

One can prove Theorems 4.1 and 4.2 by directly applying Lemma 4.5. However, we prefer to first prove (and then use) a further sufficient condition for $\mathrm{Lip}_0(M)$ to have the SSD2P, which involves only conditions on the metric of the space M and is therefore easy to handle.

Definition 4.6. We say that a metric space M has the *sequential strong long trapezoid property* (briefly, *seq-SLTP*) if, for every $\varepsilon > 0$, there exist pairwise disjoint subsets $A_1, A_2, \dots$ of M such that, for every $m \in \mathbb{N}$, there are $u_m, v_m \in A_m$ with $u_m \neq v_m$ satisfying, for all $x, y \in M \backslash A_m$,

$$(1-\varepsilon)\big(d(x,y) + d(u_m, v_m)\big) \leqslant d(x, u_m) + d(y, v_m), \tag{4.3}$$

and, for all $x, y, z, w \in M \backslash A_m$,

$$\begin{aligned} &(1-\varepsilon)\big(d(x,y) + d(z,w) + 2d(u_m, v_m)\big) \\ &\qquad\qquad \leqslant d(x, u_m) + d(y, u_m) + d(z, v_m) + d(w, v_m). \end{aligned} \tag{4.4}$$

Theorem 4.7. *Let M be a pointed metric space. If M has the seq-SLTP, then $\mathrm{Lip}_0(M)$ has the SSD2P.*

Remark 4.8. We do not know whether the converse of Theorem 4.7 holds. However, the seq-SLTP is strictly stronger than the SLTP (see Example 4.12).

Proof of Theorem 4.7. Assume that M has the seq-SLTP. Let $\delta > 0$, $n \in \mathbb{N}$, $h_1, \dots, h_n \in \mathrm{Lip}_0(M)$ with $\|h_i\| \leqslant 1 - \delta$ for every $i \in \{1, \dots, n\}$, and let $\mu \in ba(\widetilde{M})$ with only non-negative values where $\widetilde{M}$ is as in (4.1).

By the seq-SLTP, there exist subsets $A_1, A_2, \dots$ of M and points $u_m, v_m \in A_m$ as in Definition 4.6 with $\varepsilon = \delta$. By Lemma 2.4 there exist functions $f_1, \dots, f_n, g \in \mathrm{Lip}_0(M)$ satisfying

- $f_i|_N = h_i|_N$ for every $i \in \{1, \dots, n\}$;
- $g|_N = 0$ and $\|g\| \geqslant 1 - \varepsilon$;
- $\|f_i \pm g\| \leqslant 1$ for every $i \in \{1, \dots, n\}$.

By Lemma 4.5, the space $\mathrm{Lip}_0(M)$ has the SSD2P. □

Theorems 4.1 and 4.2 are immediate corollaries of Theorem 4.7 teamed with the following Propositions 4.9 and 4.10, respectively.

Proposition 4.9. *An unbounded or not uniformly discrete metric space has the seq-SLTP.*

Proposition 4.10. *For every $n \in \mathbb{N}$, the metric space K_n has the seq-SLTP.*

In the proof of Proposition 4.9, we make use of the following lemma.

Lemma 4.11. *Let $\varepsilon > 0$, $p \in M$, $0 \leqslant s < r$, and $u, v \in B(p,r) \backslash B(p,s)$ be such that*

$$4s \leqslant \varepsilon d(u,v)$$

and, for every $x \in M \backslash B(p,r)$, one has

$$2d(u,v) \leqslant \varepsilon \min\big\{d(x,u), d(x,v)\big\}.$$

Then, for all $x, y, z, w \in M \backslash A$, where $A = B(p,r) \backslash B(p,s)$, one has

$$(1-\varepsilon)\big(d(x,y) + d(u,v)\big) \leqslant d(x,u) + d(y,v) \tag{4.5}$$

and

$$\begin{aligned} (1-\varepsilon)\big(d(x,y) + d(z,w) + 2d(u,v)\big) \\ \leqslant d(x,u) + d(y,u) + d(z,v) + d(w,v). \end{aligned} \tag{4.6}$$

Proof. We first consider the inequality (4.5). Let $x, y \in M \backslash A$. If at least one of them, say x, does not belong to $B(p,r)$, then

$$\begin{aligned} d(x,u) + d(y,v) &\geqslant d(x,u) + d(y,u) - d(u,v) \\ &\geqslant (1-\varepsilon)d(x,y) + \varepsilon d(x,u) - d(u,v) \\ &\geqslant (1-\varepsilon)d(x,y) + 2d(u,v) - d(u,v) \\ &\geqslant (1-\varepsilon)\big(d(x,y) + d(u,v)\big). \end{aligned}$$

If $x, y \in B(p,r)$, then $x, y \in B(p,s)$, and therefore

$$\begin{aligned} d(x,u) + d(y,v) &\geqslant d(u,v) - d(x,y) \\ &= (1-\varepsilon)d(u,v) + \varepsilon d(u,v) - d(x,y) \\ &\geqslant (1-\varepsilon)d(u,v) + 4s - d(x,y) \\ &\geqslant (1-\varepsilon)d(u,v) + 2d(x,y) - d(x,y) \\ &\geqslant (1-\varepsilon)\big(d(u,v) + d(x,y)\big). \end{aligned}$$

We now consider the inequality (4.6). Let $x, y, z, w \in M \backslash A$. If at least one of them, say x, does not belong to $B(p,r)$, then

$$\begin{aligned} d(x,u) + d(y,u) + d(z,v) + d(w,v) &\geqslant (1-\varepsilon)\big(d(x,y) + d(z,w)\big) + \varepsilon d(x,u) \\ &\geqslant (1-\varepsilon)\big(d(x,y) + d(z,w) + 2d(u,v)\big). \end{aligned}$$

If $x, y, z, w \in B(p,r)$, then $x, y, z, w \in B(p,s)$, and therefore

$$\begin{aligned} d(x,u) + d(z,v) &\geqslant d(u,v) - d(x,z) \\ &= (1-\varepsilon)d(u,v) + \varepsilon d(u,v) - d(x,z) \\ &\geqslant (1-\varepsilon)d(u,v) + 2s \\ &\geqslant (1-\varepsilon)\big(d(u,v) + d(x,y)\big) \end{aligned}$$

and, similarly,

$$d(y,u) + d(w,v) \geqslant (1-\varepsilon)\big(d(u,v) + d(z,w)\big).$$

□

Proof of Proposition 4.9. Let M be a pointed metric space, and let $\varepsilon > 0$.

First, assume that M is unbounded. Letting $r_0 = 1$, we can inductively define points $u_m, v_m \in M \backslash B(0, r_{m-1})$ with $u_m \neq v_m$ and real numbers $r_m > 0$, $m = 1, 2, \ldots$, satisfying, for every $m \in \mathbb{N}$, the inequalities $4r_{m-1} \leqslant \varepsilon d(u_m, v_m)$, $r_m > r_{m-1}$, and

$$2d(u_m, v_m) \leqslant \varepsilon \min\big\{d(x, u_m), d(x, v_m)\big\} \quad \text{for every } x \in M\backslash B(0, r_m). \tag{4.7}$$

Now the sets $A_m \coloneqq B(0, r_m)\backslash B(0, r_{m-1})$, $m = 1, 2, \ldots$, are pairwise disjoint. For every $m \in \mathbb{N}$, Lemma 4.11 with $p = 0$, $r = r_m$, $s = r_{m-1}$, $u = u_m$, and $v = v_m$ implies that, for all $x, y, z, w \in M\backslash A_m$, the inequalities (4.3) and (4.4) hold.

Assume now that M is not uniformly discrete. We first consider the case when M has a limit point p. Starting with $r_1 = 1$, we can inductively define points $u_m, v_m \in B(p, r_m)\backslash\{p\}$ with $u_m \neq v_m$ and real numbers $r_m > 0$, $m = 1, 2, \ldots$, satisfying, for every $m \in \mathbb{N}$, the condition (4.7), $r_{m+1} < r_m$, and $4r_{m+1} \leqslant \varepsilon d(u_m, v_m)$. Now the sets $A_m \coloneqq B(p, r_m)\backslash B(p, r_{m+1})$, $m = 1, 2, \ldots$, are pairwise disjoint. For every $m \in \mathbb{N}$, Lemma 4.11 with $r = r_m$, $s = r_{m+1}$, $u = u_m$, and $v = v_m$ implies that, for all $x, y, z, w \in M\backslash A_m$, the inequalities (4.3) and (4.4) hold.

Finally, consider the case when M has no limit points. Since M is not uniformly discrete, there exist points $u_m, v_m \in M$ with $u_m \neq v_m$, $m = 1, 2, \ldots$, such that $d(u_m, v_m) \to 0$. Since M has no limit points, we may assume, after passing to subsequences if necessary, that there is an $r > 0$ such that $d(u_m, u_n) \geqslant 2r$ whenever $m, n \in \mathbb{N}$ with $m \neq n$. Furthermore, we may assume that $v_m \in B(u_m, r)$ and $4d(u_m, v_m) \leqslant \varepsilon r$ for every $m \in \mathbb{N}$. Now the sets $A_m \coloneqq B(u_m, r)$, $m = 1, 2, \ldots$, are pairwise disjoint. For every $m \in \mathbb{N}$ and every $x \in M\backslash B(u_m, r)$, one has

$$\varepsilon\, d(x, u_m) \geqslant \varepsilon\, r > 2d(u_m, v_m)$$

and

$$\varepsilon\, d(x, v_m) \geqslant \varepsilon\big(d(x, u_m) - d(u_m, v_m)\big) > \varepsilon\big(r - \tfrac{1}{2}r\big) = \tfrac{1}{2}\,\varepsilon\, r \geqslant 2d(u_m, v_m),$$

thus Lemma 4.11 with $p = u = u_m$, $s = 0$, and $v = v_m$ implies that, for all $x, y, z, w \in M\backslash A_m$, the inequalities (4.3) and (4.4) hold. □

Proof of Proposition 4.10. Let $n \in \mathbb{N}$. For every $m \in \mathbb{N}$, define

$$A_m = \big\{(x_j)_{j=1}^\infty \in K_n \colon \max_j x_j = n \text{ and } x_j < n \text{ for every } j \notin \{2m-1, 2m\}\big\},$$

$u_m = ne_{2m-1} + (n-1)e_{2m}$, and $v_m = (n-1)e_{2m-1} + ne_{2m}$. Note that the sets $A_1, A_2, \ldots$ are pairwise disjoint. Fix an $m \in \mathbb{N}$. Clearly, $u_m, v_m \in A_m$ and $d(u_m, v_m) = 1$. We show that, for all $x, y \in K_n \backslash A_m$,

$$d(x,y) + d(u_m, v_m) \leqslant d(x, u_m) + d(y, v_m),$$

and, for all $x, y, z, w \in K_n \backslash A_m$,

$$\begin{aligned} &d(x,y) + d(z,w) + 2d(u_m, v_m) \\ &\qquad \leqslant d(x, u_m) + d(y, u_m) + d(z, v_m) + d(w, v_m). \end{aligned}$$

Fix $x = (x_j)_{j=1}^\infty, y = (y_j)_{j=1}^\infty \in K_n \backslash A_m$. It suffices to show that the following inequalities hold:

$$\begin{aligned} d(x,y) + d(u_m, v_m) &\leqslant d(x, u_m) + d(y, v_m), \\ d(x,y) + d(u_m, v_m) &\leqslant d(x, u_m) + d(y, u_m), \\ d(x,y) + d(u_m, v_m) &\leqslant d(x, v_m) + d(y, v_m). \end{aligned}$$

To this end, let $j \in \mathbb{N}$ be such that $d(x,y) = |x_j - y_j|$. Without loss of generality, we assume that $x_j \geqslant y_j$. Notice that, if $j \notin \{2m-1, 2m\}$, then $x_j \geqslant d(x,y)$, and therefore $d(x, u_m) \geqslant d(x,y)$ and $d(x, v_m) \geqslant d(x,y)$. If $j \in \{2m-1, 2m\}$, then $y_j \leqslant n - 1 - d(x,y)$ because $x_j \leqslant n-1$, and hence $d(y, u_m) \geqslant d(x,y)$ and $d(y, v_m) \geqslant d(x,y)$. Since $d(u_m, v_m) = 1$, the desired inequalities hold. □

We end this section by giving an example of a metric space M with the SLTP but without the seq-SLTP. In fact, this M does not even have the seq-LTP (see Definition 4.14 below). By Theorem 2.3, the space $\mathrm{Lip}_0(M)$ has the w^*-SSD2P. It remains unknown whether $\mathrm{Lip}_0(M)$ has the SSD2P.

Example 4.12. Let $M = \{a_k, b_k, c_k \colon k \in \mathbb{N}\}$ be the metric space where, for every $k \in \mathbb{N}$,

$$d(a_k, c_k) = 2,$$

and, for all $k, l \in \mathbb{N}$ with $k < l$,

$$d(a_k, b_l) = d(b_k, b_l) = d(c_k, b_l) = 2,$$

and the distance between two different elements is 1 in all other cases (see Figure 4.1).

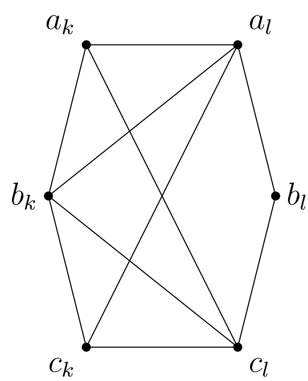

Figure 4.1: A representation of the metric space M in Example 4.12. The distances between points connected by a straight line segment are 1, the distances between other different points are 2.

We first show that the space M has the SLTP. To this end, let N be a finite subset of M. Then there exists a $K \in \mathbb{N}$ such that $N \subset \{a_k, b_k, c_k : k < K\}$. Let $u = b_K$ and $v = b_{K+1}$. Then, for every $x \in N$, one has $d(x,u) = 2$ and $d(x,v) = 2$, and therefore, for all $x, y \in N$,

$$d(x,y) + d(u,v) \leqslant 4 = d(x,u) + d(y,v),$$

and, for all $x, y, z, w \in N$,

$$d(x,y) + d(z,w) + 2d(u,v) \leqslant 8 = d(x,u) + d(y,u) + d(z,v) + d(w,v).$$

Now suppose for contradiction that M has the seq-SLTP. Then there exist pairwise disjoint subsets A_1, A_2, and A_3 of M such that, for every $m \in \{1,2,3\}$, there are $u_m, v_m \in A_m$ with $u_m \neq v_m$ such that the inequality (4.3) with $\varepsilon < 1/3$ holds for all $x, y \in M \backslash A_m$. Let $K \in \mathbb{N}$ be such that $u_1, v_1, u_2, v_2, u_3, v_3 \in \{a_k, b_k, c_k : k < K\}$. For every $m \in \{1,2,3\}$, one has

$$\big(1-\varepsilon\big)\big(d(a_K, c_K) + d(u_m, v_m)\big) \geqslant 3(1-\varepsilon) > 2 = d(a_K, u_m) + d(c_K, v_m),$$

which implies $a_K \in A_m$ or $c_K \in A_m$. It follows that A_1, A_2, and A_3 are not pairwise disjoint, a contradiction.

4.2 The SSD2P, SD2P, and D2P are different properties for spaces of Lipschitz functions

In this section, we give an example of a metric space M such that the corresponding space $\mathrm{Lip}_0(M)$ has the SD2P but fails the SSD2P, and of a metric space M such that the corresponding space $\mathrm{Lip}_0(M)$ has the D2P but fails the SD2P, thus showing that the SSD2P, SD2P, and D2P are three different properties for the spaces of Lipschitz functions. This answers an implicit question in [28, Introduction]. For these two examples, we first give sufficient conditions for the space of Lipschitz functions to have the SD2P, and the D2P, as we did for the SSD2P in the previous section. We start with an analogue of Lemma 4.5 for the SD2P.

Lemma 4.13. *Let M be a pointed metric space and let $\widetilde{M}$ be as in (4.1). Suppose that, whenever $\delta > 0$, $n \in \mathbb{N}$, $h_1, \dots, h_n \in \mathrm{Lip}_0(M)$ with $\|h_i\| \leqslant 1-\delta$ for every $i \in \{1, \dots, n\}$, and $\mu \in ba(\widetilde{M})$ with only non-negative values, there exist a subset A of M, elements $u, v \in A$ with $u \neq v$, and functions $f_1, \dots, f_n \in B_{\mathrm{Lip}_0(M)}$ satisfying*

- *$\mu(\Gamma_{1,A}) < \delta$ and $\mu(\Gamma_{2,A}) < \delta$;*
- *$f_i|_{M\setminus A} = h_i|_{M\setminus A}$ for every $i \in \{1, \dots, n\}$;*
- *$f_i(u) - f_i(v) \geqslant (1-\delta)d(u,v)$ for every $i \in \{1, \dots, n\}$.*

Then the space $\mathrm{Lip}_0(M)$ has the SD2P.

Proof. Let $n \in \mathbb{N}$, let $F_1, \dots, F_n \in S_{\mathrm{Lip}_0(M)^*}$, and let $\varepsilon > 0$. By [8, Corollary 2.2] and [21, Proposition 2.2], it suffices to find $u, v \in M$ such that $\|F_i + m_{u,v}\| \geqslant 2 - \varepsilon$ for every $i \in \{1, \dots, n\}$.

For every $i \in \{1, \dots, n\}$, let $\mu_i \in ba(\widetilde{M})$ with $|\mu_i|(\widetilde{M}) = 1$ satisfy (4.2) with F and μ replaced by F_i and μ_i, respectively. Define $\mu = |\mu_1| + \dots + |\mu_n|$. Fix a real number $\delta > 0$ satisfying $7\delta \leqslant \varepsilon$. For every $i \in \{1, \dots, n\}$, pick a function $h_i \in S(F_i, 2\delta)$ with $\|h_i\| \leqslant 1 - \delta$.

Let a subset A of M, elements $u, v \in A$ with $u \neq v$, and functions $f_1, \dots, f_n \in \mathrm{Lip}_0(M)$ satisfy the conditions in the lemma. Setting $\Gamma_A = \Gamma_{1,A} \cup \Gamma_{2,A}$, one has $\mu(\Gamma_A) < 2\delta$, hence, whenever $i \in \{1, \dots, n\}$, (observing that $\widetilde{f}_i|_{\widetilde{M}\setminus\Gamma_A} = \widetilde{h}_i|_{\widetilde{M}\setminus\Gamma_A}$)

$$\begin{aligned} F_i(f_i) = \int_{\widetilde{M}} \widetilde{f}_i \, d\mu_i &= \int_{\widetilde{M}} \widetilde{h}_i \, d\mu_i + \int_{\Gamma_A} \left(\widetilde{f}_i - \widetilde{h}_i\right) d\mu_i \\ &= F_i(h_i) + \int_{\Gamma_A} \left(\widetilde{f}_i - \widetilde{h}_i\right) d\mu_i \\ &> 1 - 2\delta - 2|\mu_i|(\Gamma_A) > 1 - 6\delta, \end{aligned}$$

and thus

$$(F_i + m_{u,v})(f_i) = F_i(f_i) + \frac{f_i(u) - f_i(v)}{d(u,v)} > 1 - 6\delta + 1 - \delta \geqslant 2 - \varepsilon.$$

$\square$

We next prove (and then use) a further sufficient condition for $\mathrm{Lip}_0(M)$ to have the SD2P—an analogue of Theorem 4.7—which involves only conditions on the metric of the space M and is therefore easy to handle.

Definition 4.14 (cf. [34, Theorem 3.1, (3)]). We say that a metric space M has the *sequential long trapezoid property* (briefly, *seq-LTP*) if, for every $\varepsilon > 0$, there exist pairwise disjoint subsets $A_1, A_2, \ldots$ of M such that, for every $m \in \mathbb{N}$, there are $u_m, v_m \in A_m$ with $u_m \neq v_m$ satisfying, for all $x, y \in M \setminus A_m$,

$$(1-\varepsilon)\big(d(x,y) + d(u_m, v_m)\big) \leqslant d(x, u_m) + d(y, v_m). \tag{4.8}$$

Clearly every metric space with the seq-SLTP has the seq-LTP as the conditions (4.3) and (4.8) are the same.

Theorem 4.15 (cf. [32, Theorem 3.1, (3)⇒(1)]). *Let M be a pointed metric space. If M has the seq-LTP, then* $\mathrm{Lip}_0(M)$ *has the SD2P.*

Remark 4.16. We do not know whether the converse of Theorem 4.15 holds. Note that the seq-LTP is strictly stronger than the LTP (see Example 4.12 above).

Proof of Theorem 4.15. Assume that M has the seq-LTP. Let $\delta > 0$, let $n \in \mathbb{N}$, let $h_1, \ldots, h_n \in \mathrm{Lip}_0(M)$ with $\|h_i\| \leqslant 1-\delta$ for every $i \in \{1, \ldots, n\}$, and let $\mu \in ba(\widetilde{M})$ with only non-negative values where $\widetilde{M}$ is as in (4.1). By Lemma 4.13, it suffices to find a subset A of M, elements $u, v \in A$ with $u \neq v$, and functions $f_1, \ldots, f_n \in \mathrm{Lip}_0(M)$ satisfying the conditions of that lemma.

By the seq-LTP, there exist subsets $A_1, A_2, \ldots$ of M and points $u_m, v_m \in A_m$, $m = 1, 2, \ldots$, as in Definition 4.14 with $\varepsilon = \delta$. Since the sets $A_1, A_2, \ldots$ are pairwise disjoint, there exists an $m \in \mathbb{N}$ such that $\mu(\Gamma_{1,A_m}) < \delta$ and $\mu(\Gamma_{2,A_m}) < \delta$. Let $A = A_m$, $u = u_m$, and $v = v_m$.

Fix $i \in \{1, \ldots, n\}$. Define the function f_i by $f_i|_{M\setminus A} = h_i|_{M\setminus A}$,

$$f_i(u) = \inf_{x \in M\setminus A} \big(f_i(x) + d(x,u)\big),$$

and

$$f_i(y) = \sup_{x \in \{u\} \cup M\setminus A} \big(f_i(x) - d(x,y)\big) \quad \text{for every } y \in A\setminus\{u\}.$$

Since $\|f_i\| \leqslant 1$, it remains to show that $f_i(u) - f_i(v) \geqslant (1-\delta)d(u,v)$. If $f_i(v) = f_i(u) - d(u,v)$, then the inequality holds. Suppose now that this is not the case. Then

$$\begin{aligned}
f_i(u) - f_i(v) &= \inf_{x,y \in M \setminus A} \big(f_i(x) + d(x,u) - f_i(y) + d(y,v) \big) \\
&\geqslant \inf_{x,y \in M \setminus A} \big(-(1-\delta) d(x,y) + d(x,u) + d(y,v) \big) \\
&\geqslant (1-\delta) d(u,v).
\end{aligned}$$

□

In Chapter 2, we gave an example of a metric space M with the LTP but without the SLTP (see Example 2.5). We now recall that metric space M to further show that it even has the seq-LTP. Therefore, $\mathrm{Lip}_0(M)$ has the SD2P but not the w^*-SSD2P. To our knowledge, this is the first known example of such a Lipschitz function space, showing that the properties SD2P and (w^*-)SSD2P are really different for the class of Lipschitz function spaces.

Example 4.17. Let $M = \{a_1, a_2, b_1, b_2\} \cup \{u_m, v_m : m \in \mathbb{N}\}$ be the metric space where, for all $i, j \in \{1, 2\}$ and all $m \in \mathbb{N}$,

$$d(a_i, b_j) = d(a_i, u_m) = d(b_i, v_m) = d(u_m, v_m) = 1,$$

and the distance between two different elements is 2 in all other cases.

We show that M has the seq-LTP. To this end, it suffices to show that, whenever $m \in \mathbb{N}$, one has, for all $x, y \in M \setminus A_m$,

$$d(x,y) + d(u_m, v_m) \leqslant d(x, u_m) + d(y, v_m),$$

where $A_m = \{u_m, v_m\}$. Fix an $m \in \mathbb{N}$ and let $x, y \notin A_m$. If $d(x, u_m) + d(y, v_m) \geqslant 3$, then the desired inequality holds because $d(u_m, v_m) = 1$. If $d(x, u_m) + d(y, v_m) = 2$, then $x \in \{a_1, a_2\}$ and $y \in \{b_1, b_2\}$, and therefore $d(x, y) = 1$; thus the desired inequality holds.

Our second aim in this section is to give an example of a metric space M such that the corresponding space $\mathrm{Lip}_0(M)$ has the D2P but fails the SD2P. For this example, we need a sufficient condition for the space $\mathrm{Lip}_0(M)$ to have the D2P. Here we do not have a condition which involves only the metric of the underlying space M as we did with Theorems 4.7 and 4.15 for the SSD2P and the SD2P, respectively. However, the following analogue of Lemmata 4.5 and 4.13 for the D2P is suitable for our purposes.

Lemma 4.18. *Let M be a pointed metric space and let $\widetilde{M}$ be as in (4.1). Suppose that, whenever $\delta > 0$, $h \in \mathrm{Lip}_0(M)$ with $\|h\| \leqslant 1 - \delta$, and $\mu \in ba(\widetilde{M})$ with only non-negative values, there exist a subset A of M, elements $u, v \in A$ with $u \neq v$, and functions $f, g \in B_{\mathrm{Lip}_0(M)}$ satisfying*

- $\mu(\Gamma_{1,A}) < \delta$ *and* $\mu(\Gamma_{2,A}) < \delta$;
- $f|_{M\setminus A} = g|_{M\setminus A} = h|_{M\setminus A}$;
- $f(u) - f(v) \geqslant (1-\delta)d(u,v)$ *and* $g(u) - g(v) \leqslant -(1-\delta)d(u,v)$.

Then the space $\mathrm{Lip}_0(M)$ *has the D2P.*

Proof of Lemma 4.18. Let $n \in \mathbb{N}$, let $F_1, \dots, F_n \in S_{\mathrm{Lip}_0(M)^*}$, let $\varepsilon > 0$, and let $\phi \in B_{\mathrm{Lip}_0(M)}$. It suffices to find $f, g \in \mathrm{Lip}_0(M)$ with $\|f\| \leqslant 1$ and $\|g\| \leqslant 1$ such that $|F_i(f-\phi)| < \varepsilon$ and $|F_i(g-\phi)| < \varepsilon$ for every $i \in \{1, \dots, n\}$, and $\|f - g\| > 2 - \varepsilon$.

For every $i \in \{1, \dots, n\}$, let $\mu_i \in ba(\widetilde{M})$ with $|\mu_i|(\widetilde{M}) = 1$ satisfy (4.2) with F and μ replaced by F_i and μ_i, respectively. Define $\mu = |\mu_1| + \dots + |\mu_n|$. Fix a real number $\delta > 0$ satisfying $5\delta \leqslant \varepsilon$. Let $h = (1-\delta)\phi$.

Let a subset A of M, elements $u, v \in A$ with $u \neq v$, and functions $f, g \in \mathrm{Lip}_0(M)$ satisfy the conditions in the lemma. Setting $\Gamma_A = \Gamma_{1,A} \cup \Gamma_{2,A}$, one has $\mu(\Gamma_A) < 2\delta$, hence, for every $i \in \{1, \dots, n\}$ (observing that $\widetilde{f}|_{\widetilde{M}\setminus\Gamma_A} = \widetilde{h}|_{\widetilde{M}\setminus\Gamma_A}$),

$$\begin{aligned}|F_i(f-\phi)| &\leqslant |F_i(f-h)| + \delta = \left|\int_{\widetilde{M}} \big(\widetilde{f} - \widetilde{h}\big)\, d\mu_i\right| + \delta = \left|\int_{\Gamma_A} \big(\widetilde{f} - \widetilde{h}\big)\, d\mu_i\right| + \delta \\ &\leqslant 2|\mu_i|(\Gamma_A) + \delta < 5\delta \leqslant \varepsilon.\end{aligned}$$

Similarly, $|F_i(g-\phi)| < \varepsilon$ for every $i \in \{1, \dots, n\}$. It remains to observe that

$$\begin{aligned}\|f-g\| &\geqslant \frac{(f-g)(u) - (f-g)(v)}{d(u,v)} = \frac{f(u) - f(v) - g(u) + g(v)}{d(u,v)} \\ &\geqslant \frac{2(1-\delta)\, d(u,v)}{d(u,v)} = 2(1-\delta) > 2 - \varepsilon.\end{aligned}$$

□

We now introduce a space M for which the corresponding Lipschitz function space $\mathrm{Lip}_0(M)$ has the D2P but not the (w^*-)SD2P. To our knowledge, this is the first such example in the class of Lipschitz function spaces.

EXAMPLE 4.19. Let $M = \big\{a_i, u_m^i, v_m^i \colon i \in \{1,2,3\}, m \in \mathbb{N}\big\}$ be the metric space where, for all $i, j \in \{1,2,3\}$ with $i \neq j$ and all $m \in \mathbb{N}$,

$$d(a_i, u_m^j) = d(a_i, v_m^j) = 1,$$

and, for all $j \in \{1,2,3\}$ and all $m \in \mathbb{N}$,

$$d(u_m^j, v_m^j) = 1,$$

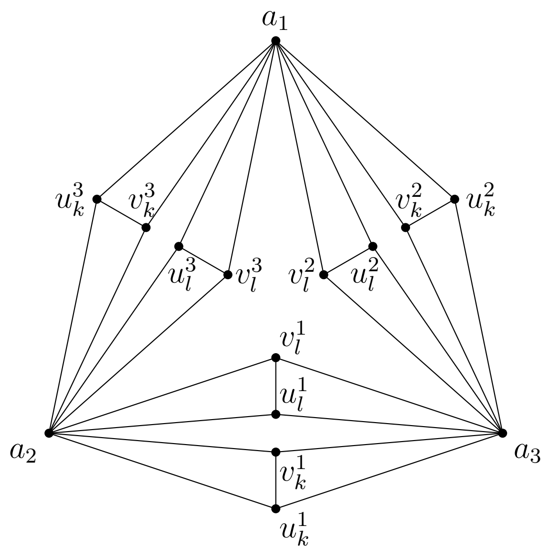

Figure 4.2: A representation of the metric space M in Example 4.19. The distances between points connected by a straight line segment are 1, the distances between other different points are 2.

and the distance between two different elements is 2 in all other cases (see Figure 4.2).

We first show that the space $\mathrm{Lip}_0(M)$ has the D2P. We make use of Lemma 4.18. Let $\delta > 0$, let $h \in \mathrm{Lip}_0(M)$ with $\|h\| \leqslant 1$, and let $\mu \in ba(\widetilde{M})$ with only non-negative values. We may assume that $h(a_1) \leqslant h(a_2) \leqslant h(a_3)$. Set $L = \inf h(M)$. If $h(a_2) \leqslant L + 1$, then let $k = 3$ and $c = L$; otherwise, let $k = 1$ and $c = L + 1$. Choose an $m \in \mathbb{N}$ so that $\mu(\Gamma_{1,A}) < \delta$ and $\mu(\Gamma_{2,A}) < \delta$ where $A = \{u_m^k, v_m^k\}$. Let $u = u_m^k$ and $v = v_m^k$, and define $f, g \colon M \to \mathbb{R}$ by

$$f(x) = \begin{cases} h(x) & \text{if } x \in M \backslash A; \\ c + 1 & \text{if } x = u; \\ c & \text{if } x = v, \end{cases} \qquad \text{and} \qquad g(x) = \begin{cases} h(x) & \text{if } x \in M \backslash A; \\ c & \text{if } x = u; \\ c + 1 & \text{if } x = v. \end{cases}$$

Then

$$f(u) - f(v) = g(v) - g(u) = 1 = d(u, v).$$

It is straightforward to verify that $\|f\| = 1$ and $\|g\| = 1$. By Lemma 4.18, $\mathrm{Lip}_0(M)$ has the D2P.

We now show that the space $\mathrm{Lip}_0(M)$ does not have the w^*-SD2P. It suffices to show that space M does not have the LTP. Let $N = \{a_1, a_2, a_3\}$ and let $\varepsilon < 1/3$. Whenever $u, v \in M$ with $u \neq v$, there exist $x, y \in N$ with

$x \neq y$ such that $d(x,u) \leqslant 1$ and $d(y,v) \leqslant 1$. Since $d(x,y) = 2$, one has

$$(1-\varepsilon)\big(d(u,v) + d(x,y)\big) \geqslant 3(1-\varepsilon) > 2 \geqslant d(x,u) + d(y,v).$$

4.3 Local norm-one Lipschitz function is a Daugavet-point

Let M be a pointed metric space. In this section, we show that certain norm-one elements f of $\mathrm{Lip}_0(M)$ are *Daugavet-points*, i.e, given a slice S of the unit ball of $\mathrm{Lip}_0(M)$ and an $\varepsilon > 0$, there exists a $g \in S$ with $\|f - g\| > 2 - \varepsilon$.

Definition 4.20 (see [25, Definition 2.5])**.** A function $f \in \mathrm{Lip}_0(M)$ is said to be *local* if, for every $\varepsilon > 0$, there are $u, v \in M$ with $u \neq v$ such that $d(u,v) < \varepsilon$ and $f(m_{u,v}) > \|f\| - \varepsilon$.

The question of whether every local f in the unit sphere of $\mathrm{Lip}_0(M)$ is a Daugavet point was posed and addressed in [25]; there, it was shown that

1. every local f on the unit sphere of $\mathrm{Lip}_0(M)$ is a weak* Daugavet-point, i.e., given a weak* slice S of the unit ball of $\mathrm{Lip}_0(M)$ and an $\varepsilon > 0$, there exists $g \in S$ with $\|f - g\| > 2 - \varepsilon$;

2. every spreadingly local f (see [25, Definition 2.5]) in the unit sphere of $\mathrm{Lip}_0(M)$ is a Daugavet-point.

The result (2) is yielded as a local argument of [24, Theorem 3.1] which says that if M is spreadingly local, then $\mathrm{Lip}_0(M)$ has the Daugavet property. The way we look at $\mathrm{Lip}_0(M)^*$ allows us to improve upon all these results and fully answer the aforementioned question with Theorem 4.4. The latter is, in fact, a straightforward consequence of the following result.

Proposition 4.21 (cf. [25, Theorem 2.6])**.** *Let M be a pointed metric space, let $F \in S_{\mathrm{Lip}_0(M)^*}$, and let (u_n) and (v_n) be two sequences of elements in M such that $u_n \neq v_n$ for every $n \in \mathbb{N}$. If $d(u_n, v_n) \to 0$, then $\|F + m_{u_n,v_n}\| \to 2$.*

In the proof of Proposition 4.21, we shall repeatedly make use of the following simple lemma.

Lemma 4.22. *Let M be a metric space, let $u, x, y \in M$, and let r and θ be real numbers with $r > 0$ and $0 < \theta < 1$. Suppose that $y \in B(u, \theta r)$ and $x \in M \backslash B(u, r)$. Then $d(y,u) \leqslant \frac{\theta}{1-\theta}\, d(x,y)$.*

Proof. Observe that $d(x,y) \geqslant (1-\theta)r$ because otherwise one would have

$$d(x,u) \leqslant d(x,y) + d(y,u) < (1-\theta)r + \theta r = r,$$

a contradiction. It follows that $d(y,u) < \theta r \leqslant \frac{\theta}{1-\theta}\, d(x,y)$, as desired. $\square$

Proof of Proposition 4.21. Assume that $d(u_n, v_n) \to 0$. Let $\varepsilon > 0$. Our aim is to prove that there exists an $n \in \mathbb{N}$ such that $\|F + m_{u_n,v_n}\| > 2-\varepsilon$. To that end, fix a real number $\delta > 0$ satisfying $7\delta \leqslant \varepsilon$ and $h \in S(F, 2\delta)$ with $\|h\| \leqslant 1-\delta$. Let $\widetilde{M}$ be as in (4.1), and let $\mu \in ba(\widetilde{M})$ with $|\mu|(\widetilde{M}) = 1$ satisfy (4.2). It suffices to show that there is a subset A of M with $|\mu(\Gamma_{1,A})| < \delta$ and $|\mu(\Gamma_{2,A})| < \delta$ such that, for some $n \in \mathbb{N}$, there exists a function $f \in B_{\mathrm{Lip}_0(M)}$ such that $f|_{M\setminus A} = h|_{M\setminus A}$ and $f(u_n) - f(v_n) \geqslant (1-\delta)d(u_n, v_n)$. Indeed, suppose that such A, n, and f have been found. Then, setting $\Gamma_A = \Gamma_{1,A} \cup \Gamma_{2,A}$, one has $|\mu|(\Gamma_A) < 2\delta$, and thus (observing that $\widetilde{f}|_{\widetilde{M}\setminus\Gamma_A} = \widetilde{h}|_{\widetilde{M}\setminus\Gamma_A}$)

$$\begin{aligned} F(f) &= \int_{\widetilde{M}} \widetilde{f}\, d\mu = \int_{\widetilde{M}} \widetilde{h}\, d\mu + \int_{\Gamma_A} \big(\widetilde{f} - \widetilde{h}\big)\, d\mu \\ &= F(h) + \int_{\Gamma_A} \big(\widetilde{f} - \widetilde{h}\big)\, d\mu \\ &> 1 - 2\delta - 2|\mu|(\Gamma_A) \geqslant 1 - 6\delta, \end{aligned}$$

and, therefore,

$$\|F + m_{u_n,v_n}\| \geqslant F(f) + \frac{f(u_n) - f(v_n)}{d(u_n, v_n)} > 2 - 7\delta \geqslant 2 - \varepsilon.$$

It remains to find the A, n, and f as above. To this end, choose a real number $\theta \in (0,1)$ satisfying $\frac{\theta}{1-\theta} < \frac{\delta}{2}$. Without loss of generality, one may assume that one of the following (mutually exclusive) conditions holds:

(1) no subsequence of the sequence $(u_n)_{n=1}^{\infty}$ converges;

(2) there is a $u \in M$ such that $u_n = u$ for every $n \in \mathbb{N}$;

(3) there is a $u \in M$ with $u_n \neq u$ and $v_n \neq u$ for every $n \in \mathbb{N}$ such that $u_n \to u$.

(1). In this case, by passing to a subsequence, one may assume that there is an $r > 0$ such that the open balls $A_n := B(u_n, r)$, $n = 1, 2, \ldots$, are pairwise disjoint. It follows that there is an $N \in \mathbb{N}$ such that, for every $n \geqslant N$, one has $|\mu|(\Gamma_{1,A_n}) < \delta$ and $|\mu|(\Gamma_{2,A_n}) < \delta$. Pick an $n \geqslant N$ so that $d(u_n, v_n) < \theta r$. One may assume that $0 \notin A_n$. Let $A = A_n$ and define the function f by

$f|_{M\setminus A} = h|_{M\setminus A}$, $f(u_n) = h(u_n)$, $f(v_n) = h(u_n) - (1-\delta)d(u_n, v_n)$, and by extending the definition norm-preservingly to the whole M.

Then $\|f\| \leqslant 1$ because, whenever $x \in M\setminus A$, one has (taking into account that $d(u_n, v_n) \leqslant \frac{\delta}{2}\, d(x, v_n)$ by Lemma 4.22)

$$\begin{aligned} |f(x) - f(v_n)| &= |h(x) - h(u_n) + (1-\delta)\, d(u_n, v_n)| \\ &\leqslant (1-\delta)\, d(x, u_n) + (1-\delta)\, d(u_n, v_n) \\ &\leqslant (1-\delta)\big(d(x, v_n) + 2\, d(u_n, v_n)\big) \leqslant d(x, v_n). \end{aligned}$$

(2). Pick an $n \in \mathbb{N}$ and $r, s > 0$ so that $d(v_n, u) < \theta r$, $s < \theta d(v_n, u)$, and $|\mu|(\Gamma_{1,A}) < \delta$ and $|\mu|(\Gamma_{2,A}) < \delta$ where $A = B(u,r)\setminus B(u,s)$. One may assume that $0 \notin B(u,r)\setminus\{u\}$. Letting A be as above, define the function f by $f|_{M\setminus A} = h|_{M\setminus A}$, $f(v_n) = h(u) - (1-\delta)\, d(u, v_n)$, and by extending the definition norm-preservingly to the whole M. One has $\|f\| \leqslant 1$. In fact, if $x \in B(u,s)$, then (taking into account that $d(x,u) \leqslant \frac{\delta}{2}\, d(x, v_n)$ by Lemma 4.22)

$$\begin{aligned} |f(x) - f(v_n)| &= |h(x) - h(u) + (1-\delta)\, d(u, v_n)| \\ &\leqslant (1-\delta)\, d(x, u) + (1-\delta)\, d(u, v_n) \\ &\leqslant (1-\delta)\big(2d(x, u) + d(x, v_n)\big) \leqslant d(x, v_n); \end{aligned}$$

if $x \in M\setminus B(u,r)$, then, keeping in mind that $u = u_n$, the desired inequality $|f(x) - f(v_n)| \leqslant d(x, v_n)$ is obtained as in the case (1).

(3). Pick an $n \in \mathbb{N}$ and $r, s > 0$ so that

$$\max\{d(u, u_n), d(u, v_n)\} < \theta r, \quad s < \theta \min\{d(u, u_n), d(u, v_n)\},$$

and $|\mu|(\Gamma_{1,A}) < \delta$ and $|\mu|(\Gamma_{2,A}) < \delta$ where $A = B(u,r)\setminus B(u,s)$. Letting A be as above, define the function f by $f|_{M\setminus A} = h|_{M\setminus A}$, $f(u_n) = h(u) + (1-\delta)\, d(u, u_n)$, $f(v_n) = f(u_n) - (1-\delta)\, d(u_n, v_n)$, and by extending the definition norm-preservingly to the whole M. By calculations similar to those performed in the cases (1) and (2), one establishes that $\|f\| \leqslant 1$. □

Recall that a metric space M is *local* if for every $\varepsilon > 0$ and for every Lipschitz function $f\colon M \to \mathbb{R}$ there are two distinct points $u, v \in M$ such that $d(u,v) < \varepsilon$ and $f(m_{u,v}) > \|f\| - \varepsilon$. The following result is an immediate consequence of the previous theorem.

Corollary 4.23 (see [13, Proposition 3.4 and Theorem 3.5], cf. [24, Theorem 3.1])**.** *Let M be a pointed metric space. If M is local, then* $\mathrm{Lip}_0(M)$ *has the Daugavet property.*

Remark 4.24. The converse statement of this result also holds (see [24, Proposition 2.3 and the remark following its proof] and [13, Theorem 3.5]). On the other hand, the converse statement of our Theorem 4.4 does not hold since there exist uniformly discrete metric spaces M for which $\mathrm{Lip}_0(M)$ has Daugavet-points. For example, let M be an infinite pointed metric space where the distance between two different elements is 1 if one of the elements is the fixed point 0, and the distance between two different elements is 2 in all other cases. Then the norm-one element $f \in \mathrm{Lip}_0(M)$, given by $f(x) = 1$ for every $x \in M\backslash\{0\}$, is a Daugavet-point.

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Diameeter-2 omadused Lipschitzi funktsiooniruumides

Kokkuvõte

Käesoleva väitekirja põhieesmärk on laiendada teadmisi diameeter-2 omaduste kohta Lipschitzi funktsiooniruumides. Diameeter-2 omadused on Banachi ruumidel vaadeldavate teatud omaduste koondnimetus. Neile omadustele on iseloomulik, et ruumi kinnise ühikkera kõik kindlat liiki osahulgad, näiteks ühikkera viilud või mittetühjad suhteliselt nõrgalt lahtised osahulgad, on diameetriga kaks nagu ühikkera ise. Vaadeldavate diameeter-2 omadusega Banachi ruum on tingimata lõpmatumõõtmeline, sest teadaolevalt on iga lõplikumõõtmelise Banachi ruumi ühikkeral Radon–Nikodými omadus, mistõttu sisaldab sellise ruumi ühikkera kuitahes väikese diameetriga viile.

Meetriliste ruumide vahel tegutsevad Lipschitzi funktsioonid on kõige loomulikum mitte-lineaarne analoog normeeritud ruumide vahel tegutsevatele pidevatele lineaarsetele operaatoritele. Iga Lipschitzi kujutusega on seotud kindel mittenegatiivne reaalarv, selle kujutuse Lipschitzi konstant, mis iseloomustab selle kujutuse maksimaalset võimalikku suhtelist muutust. Erilist huvi pakuvad reaalarvuliste väärtustega Lipschitzi funktsioonid. Kõik fikseeritud meetrilisest ruumist M reaalarvude ruumi tegutsevad Lipschitzi kujutused moodustavad täieliku poolnormiga vektorruumi, mida standardse samastamise võttega saab vaadelda Banachi ruumina, nn Lipschitzi funktsiooniruumina $\mathrm{Lip}_0(M)$, kus Lipschitizi kujutuse norm on võrdne tema Lipschitzi konstandiga. Osutub, et Lipschitzi funktsiooniruum on koguni kaasruum. Tema teatud täielikku eelruumi nimetatakse Lipschitzi-vabaks ruumiks (üle algselt lähtehulgaks olnud meetrilise ruumi M) ning tähistatakse sümboliga $\mathcal{F}(M)$, see Banachi ruum sisaldab loomulikul viisil algset meetrilist ruumi M. Mis tahes meetriliste ruumide vahel tegutsev Lipschitzi kujutus on jätkatav pidevaks lineaarseks operaatoriks vastavate Lipschitzi-vabade ruumide vahel, kusjuures nii, et jätku norm on täpselt algse kujutuse Lipschitzi konstant. See võimaldab (mittelineaarse) Lipschitzi funktsiooni asemel vaadelda pidevat lineaarset operaatorit, mis tegutseb Lipschitzi-vabade ruumide vahel.

Töös vaadeldakse diameeter-2 omadusi Lipschitzi funktsiooniruumides. Lipschitzi funktsiooniruumi $\mathrm{Lip}_0(M)$ $*$-nõrgale sümmeetrilisele tugevale diameeter-2 omadusele antakse kirjeldus nii vastava meetrilise ruumi M omaduse kui ka Lipschitzi-vaba ruumi $\mathcal{F}(M)$ omaduse kaudu. Näidatakse, et

Lipschitzi funktsiooniruumis on sümmeetriline tugev diameeter-2 omadus, tugev diameeter-2 omadus ja diameeter-2 omadus kõik üksteisest erinevad omadused. Lisaks näidatakse, et iga lokaalne punkt Lipschitzi funktsiooniruumis on Daugaveti punkt.

Väitekiri koosneb neljast peatükist. Esimeses peatükis esitatakse ülevaade töös vaadeldavate põhiomaduste ajaloolisest taustast, töö kokkuvõte ja töös vajalikud tähistused ning antakse väitekirja põhiosa mõistmiseks vajalikud eelteadmised.

Teises peatükis antakse Lipschitzi funktsiooniruumi $\mathrm{Lip}_0(M)$ $*$-nõrga sümmeetrilise tugeva diameeter-2 omaduse kirjeldus ruumi M omaduse kaudu. Esitatakse näide meetrilisest ruumist M, mille korral Lipschitzi funktsiooniruumil $\mathrm{Lip}_0(M)$ on $*$-nõrk tugev diameeter-2 omadus, kuid ei ole $*$-nõrka sümmeetrilist tugevat diameeter-2 omadust. Sellega näidatakse, et need omadused on Lipschitzi funktsiooniruumidel erinevad ning vastatakse artiklis [20] esitatud küsimusele. Peatükk põhineb artiklil [32].

Kolmandas peatükis antakse Lipschitzi funktsiooniruumi $\mathrm{Lip}_0(M)$ $*$-nõrga sümmeetrilise tugeva diameeter-2 omaduse kirjeldus Lipschitzi-vaba ruumi $\mathcal{F}(M)$ omaduse kaudu. Selleks defineeritakse Banachi ruumidel tükeldatava oktaeedrilisuse mõiste. Näidatakse, et Banachi ruum on tükeldatavalt oktaeedriline, kui tema kaasruumil on $*$-nõrk sümmeetriline tugev diameeter-2 omadus. Esitatakse tarvilikud ja piisavad tingimused selleks, et kahe Banachi ruumi absoluutne summa oleks tükeldatavalt oktaeedriline, ning näidatakse, et kõigi pidevate funktsioonide ruum $C(K)$ kompaktse Hausdorffi ruumi K korral pole kunagi tükeldatavalt oktaeedriline. Peatükk põhineb artiklil [33].

Neljandas peatükis näidatakse de Leeuw' teisendust kasutades, et sümmeetriline tugev diameeter-2 omadus, tugev diameeter-2 omadus ja diameeter-2 omadus on Lipschitzi funktsiooniruumide klassis erinevad omadused. Näidatakse, et ruumil $\mathrm{Lip}_0(K_n)$ on sümmeetriline tugev diameeter-2 omadus iga naturaalarvu n korral. Mainitud tulemused annavad vastused artiklis [28] esitatud kahele küsimusele. Lisaks näidatakse, et Lipschitzi funktsiooniruumi ühiksfääri iga lokaalne punkt on Daugaveti punkt. Sellega antakse vastus artiklis [25] esitatud küsimusele. Peatükk põhineb artiklil [22].

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Curriculum vitae

Name: Andre Ostrak
Date and place of birth: 07.01.1994, Sihva, Estonia
Citizenship: Estonian
Address: Institute of Mathematics and Statistics, Narva mnt 18, 51009 Tartu, Estonia
E-mail: andre.ostrak@ut.ee
Education:

2009–2012	Nõo High School
2012–2016	University of Tartu, bachelor studies in mathematics, Bachelor of Science (Mathematics), cum laude, 2016
2016–2018	University of Tartu, master studies in mathematics, Master of Science (Mathematics and Statistics), cum laude, 2018
2018–	University of Tartu, doctoral studies in mathematics

Languages: Estonian, English
Employment:

09/2018–	Cybernetica AS, junior researcher
09/2021–	Tartu Ülikool, junior research fellow of mathematics (0.75 p)

Curriculum vitae

NIMI: Andre Ostrak
SÜNNIAEG JA -KOHT: 07.01.1994, Sihva, Eesti
KODAKONDSUS: Eesti
AADRESS: TÜ matemaatika ja statistika instituut, Narva mnt 18,
51009 Tartu, Eesti
E-POST: andre.ostrak@ut.ee
HARIDUS:

2009–2012	Nõo Reaalgümnaasium
2012–2016	Tartu Ülikool, matemaatika bakalaureuseõpe, Loodusteaduse bakalaureus (matemaatika), *cum laude*, 2016
2016–2018	Tartu Ülikool, matemaatika magistriõpe, Loodusteaduse magister (matemaatika ja statistika), *cum laude*, 2018
2018–	Tartu Ülikool, matemaatika doktoriõpe

KEELTEOSKUS: eesti, inglise
TEENISTUSKÄIK:

09/2018–	Cybernetica AS, nooremteadur
09/2021–	Tartu Ülikool, matemaatika nooremteadur (0,75 k)

List of original publications

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